1. Prove that the edges of a bipartite graph with maximum degree $\Delta$ can be colored with $\Delta$ colors such that no two edges that share a vertex have the same color.

2. A square matrix $A \in \mathbb{R}^{n \times n}$ is **doubly stochastic** if the entries of the matrix are nonnegative, and the sum of entries in every row and column is equal to one. The Birkhoff-von Neumann theorem states that one can write any doubly stochastic matrix as a convex combination of permutation matrices. Prove this theorem, and furthermore show that it suffices to use a convex combination of at most $n^2 - n$ permutation matrices.

3. An **independent set** $S$ in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in $S$. If we let $P$ denote the convex hull of all (incidence vectors of) independent sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for $P$.

   (a) Give a graph $G$ for which $P$ is not equal to
   
   \[
   \{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i, j) \in E, \quad x_i \geq 0 \quad \text{for all } i \in V\}
   \]

   (b) Show that if the graph $G$ is bipartite then $P$ equals
   
   \[
   \{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i, j) \in E, \quad x_i \geq 0 \quad \text{for all } i \in V\}.
   \]

4. (Echenique, Immorlica, Vazirani 1.15) Consider an instance of the stable matching problem with $n$ doctors and $n$ hospitals, each with capacity 1. Assume there are an odd number, $k$, of stable matchings. For each doctor $d$, order his or her $k$ matches (with repetitions) according to his or her preference list and do the same for every hospital $h$. Consider assigning every doctor to the median hospital in his or her list. In this problem, we will prove that this is a stable matching.

   To do so, first let $\mu_1, \ldots, \mu_l$ be any $l$ stable matchings. For each doctor-hospital pair $(d, h)$, let $n(d, h)$ be the number of matchings in $\{\mu_1, \ldots, \mu_l\}$ where $d$ is matched to $h$. Define $x_{dh} := (1/l) \cdot n(d, h)$. Show that $x$ is a feasible solution to the fractional stable matching LP. For any $k$ with $1 \leq k \leq l$, let $\theta = (k/l) - \epsilon$ where $\epsilon > 0$ is smaller than $1/l$. Consider the stable matching $\mu_\theta$ formed by “rounding” the fractional stable matching $x$ with $\theta$ via the procedure in class. Show that $\mu_\theta$ matches each doctor $d$ to the $k^{th}$ hospital in his or her ordered list of the $l$ firms $d$ is matched to in $\{\mu_1, \ldots, \mu_l\}$. Show similarly that hospital $h$ is matched to the $(l - k + 1)^{th}$ doctor in its list.

   Use this to show that the aforementioned “median assignment” forms a stable matching.

5. Form a project team (ideally with a group of 2-3 students). Please list your teammates and write a few paragraphs about the topic you plan to work on.

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1See also Echenique/Immorlica/Vazirani Chapter 1, Section 5.1