MS&E 319 Approximation Algorithms
Instructor: Amin Saberi (saberi@stanford.edu)
Homework #4

1. Denote the second eigenvalue of the Laplacian of $G(V,E)$ by $\lambda_2(G)$. Prove that for every $S \subseteq V$, $\lambda_2(G) \leq \lambda_2(G \setminus S) + |S|$. Use that to bound the second eigenvalue by vertex connectivity.

2. Show that for any symmetric matrix X and any integer $k \geq 1$ the sum of the k largest eigenvalues of X is a convex function of X.

3. Remember a spanning tree T is α-thin for graph $G(V,E)$ if and only if for every cut $(S, \bar{S})$

$$E_T(S, \bar{S}) \leq \alpha E_G(S, \bar{S}).$$

   Prove that a k-dimensional hypercube has an $O(1/k)$-thin spanning tree. You can follow the following line of reasoning or find an original proof.

   - Suppose G has a set of cycles $C_1, C_2, \cdots, C_k$ such that (i) each cycle has exactly one edge of T, and each edge of T is an at least β cycles. (ii) Each edge not in T is in at most α cycles. Show that T is α/β-thin.
   - The next step to construct a connected thin subgraph. Decompose H_{2k} into 2^k subcubes $H(x)$ for $0 \leq x < 2^k$ each uniquely determined by the first k bits. Decompose each $H(x)$ into $k/2$ edge disjoint Hamiltonian paths and choose one from each decomposition in a way that their union is thin for H_{2k}.
   - Repeat the same for the last k digits and show the union is connected.

4. For A an $n^2 \times n^2$ symmetric matrix, we let P_A be the degree 4 polynomial

$$P_A(x) = \sum_{1 \leq i,j,k,l \leq n} A_{i,j,k,l} x_i x_j x_k x_l.$$

   We say that $A \sim B$ if $P_A = P_B$

   - Show that the set of B such that $B \sim A$ is an affine subspace of R^{n^2} i.e., it is defined by linear equations on the coefficients.
   - Prove that P_A is a sum of squares polynomial if and only if there exists a positive semidefinite matrix B such that $B \sim A$.
   - (harder, extra credit) Prove that P_A is a sum of squares polynomial if and only if there does not exist an $n^2 \times n^2$ matrix X such that for every permutation $\pi : \{1,2,3,4\} \to \{1,2,3,4\}$ and $1 \leq i_1, \cdots, i_4 \leq n$, $X_{i_1,i_2,i_3,i_4} = X_{i_{\pi(1)}, i_{\pi(2)}, i_{\pi(3)}, i_{\pi(4)}}$, X is positive semidefinite, and $tr(AX) < 0$. (Hint: this is semidefinite duality)