1. Denote the second eigenvalue of the Laplacian of $G(V, E)$ by $\lambda_2(G)$. Prove that for every $S \subseteq V$, $\lambda_2(G) \leq \lambda_2(G \setminus S) + |S|$. Use that to bound the second eigenvalue by vertex connectivity.

2. Show that for any symmetric matrix $X$ and any integer $k \geq 1$ the sum of the $k$ largest eigenvalues of $X$ is a convex function of $X$.

3. Remember a spanning tree $T$ is $\alpha$-thin for graph $G(V, E)$ if and only if for every cut $(S, \bar{S})$

$$E_T(S, \bar{S}) \leq \alpha E_G(S, \bar{S}).$$

Prove that a $k$-dimensional hypercube has an $O(1/k)$-thin spanning tree. You can follow the following line of reasoning or find an original proof.

- Suppose $G$ has a set of cycles $C_1, C_2, \ldots, C_k$ such that (i) each cycle has exactly one edge of $T$, and each edge of $T$ is an at least $\beta$ cycles. (ii) Each edge not in $T$ is in at most $\alpha$ cycles. Show that $T$ is $\alpha/\beta$-thin.

- The next step to construct a connected thin subgraph. Decompose $H_{2k}$ into $2^k$ subcubes $H(x)$ for $0 \leq x < 2^k$ each uniquely determined by the first $k$ bits. Decompose each $H(x)$ into $k/2$ edge disjoint Hamiltonian paths and choose one from each decomposition in a way that their union is thin for $H_{2k}$.

- Repeat the same for the last $k$ digits and show the union is connected.

4. For $A$ an $n^2 \times n^2$ symmetric matrix, we let $P_A$ be the degree 4 polynomial

$$P_A(x) = \sum_{1 \leq i,j,k,l \leq n} A_{i,j,k,l}x_i x_j x_k x_l.$$ 

We say that $A \sim B$ if $P_A = P_B$

- Show that the set of $B$ such that $B \sim A$ is an affine subspace of $R^{n^2}$ i.e., it is defined by linear equations on the coefficients.

- Prove that $P_A$ is a sum of squares polynomial if and only if there exists a positive semidefinite matrix $B$ such that $B \sim A$.

- (harder, extra credit) Prove that $P_A$ is a sum of squares polynomial if and only if there does not exist an $n^2 \times n^2$ matrix $X$ such that for every permutation $\pi : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ and $1 \leq i_1, \ldots, i_4 \leq n$, $X_{i_1, i_2, i_3, i_4} = X_{\pi(1), \pi(2), \pi(3), \pi(4)}$, $X$ is positive semidefinite, and $tr(AX) < 0$. (Hint: this is semidefinite duality)