In this lecture we describe combinatorial and polyhedral formulations of perfect matchings in bipartite graphs.

1 Combinatorial formulation of Matching in Bipartite Graphs

Recall Hall’s theorem that gives a necessary and sufficient condition for a bipartite graph to have a perfect matching.

**Theorem 1 (Hall’s Theorem)** A bipartite graph $G(A, B, E)$, where $|A| = |B| = n$ has a perfect matching if and only if:

$$\forall S \subseteq A, \quad |N(S)| \geq |S|.$$  

where $N(S)$ is the set of neighbors of vertices of $S$ in $B$.

This gives an efficient certificate for the statement $G$ has no perfect matching (by giving the set $S$ with $|N(S)| < |S|$), so perfect matching is in NP \cap coNP.

**Definition 2 (Augmenting Path)** A path $v_1, v_2, \ldots, v_{2k+1}$ is an augmenting path in $G(V, E)$ w.r.t a matching $M$ if and only if:

- $(v_i, v_{i+1}) \in E$ for all $i \in [2k]$, and
- $(v_{2i}, v_{2i+1}) \in M$ for all $i \in k$, and $v_1$ and $v_{2k+1}$ are not matched (they are not adjacent to any edge in $M$).

**Proposition 3** Either $M$ is a maximum matching or there exists an augmenting path in $G$ with respect to $M$.

**Proof:** If there is an augmenting path, then by swapping the edges of $M$ in the path with the edges not in $M$ in the path, we obtain a larger matching. Hence, if $M$ is maximum then there is no augmenting path.

Conversely, suppose that $M$ is not maximum. Let $N$ be another matching such that $|N| > |M|$. Consider the subgraph of $G$ formed by the symmetric difference of $N$ and $M$. Each vertex in this graph is of degree at most 2, so the graph is a union of even cycles (since $G$ is bipartite) and paths. But, there are the same number of edges from $M$ and $N$ in each cycle and even-length path. Since, $|N| > |M|$ there must exist an odd path, such that its starting and ending edges are in $N$. This is an augmenting path w.r.t $M$. 

By the previous proposition, to find a maximum matching in a bipartite graph it is enough to start with any matching and iteratively find an augmenting path in the graph to increase the size of the matching (by swapping the matching edges in the augmenting path with the edges in the augmenting path that are not in the matching).
We have thus reduced the task of finding maximum matchings to finding augmenting paths.

**Algorithm 1:** Find an augmenting path

Direct edges in $M$ from $A$ to $B$;
Direct edges in $E \setminus M$ from $B$ to $A$;
Add vertices $s, t$. Connect $s$ to all unmatched vertices of $B$. Connect all unmatched
vertices of $A$ to $t$;
Run BFS algorithm to find shortest path from $s$ to $t$.

By construction of the directed graph in the algorithm, if there exists a path from $s$ to $t$, there
exists an augmenting path. Looking at the tree produced at the last step of algorithm one can see
matchings and vertex covers are intimately related.

**Definition 4 (Vertex Cover)** A set $S \subseteq V$ is a vertex cover if for every $(u, v) \in E$, $u \in S$ or
$v \in S$.

**Lemma 5** If $M$ is a matching and $S$ is vertex cover then $|S| \geq |M|$.

**Proof:** For a given vertex cover $S$, at least one vertex of each edge of $M$ must be in $S$. So $|S| \geq |M|$.

**Theorem 6 (Konig)** If $M^*$ is a maximum matching in a bipartite graph $G$ and $S^*$ is a minimum
vertex cover in $G$ then $|M^*| = |S^*|$.

**Proof:** Given a maximum matching $M^*$, we grow an alternating graph starting from all unmatched
vertices in $A$ as layer 0. Then layer 1 consists of all neighbors of $A$. All vertices in layer 1 must be
matched, as otherwise we find an augmenting path, so we let layer 2 consist of vertices matched
to vertices in layer 1. We then let layer 3 consist of neighbors of vertices in layer 2, which must
again be matched since there is no augmenting path. Assuming the graph $G$ is connected, we can
continue this process until all vertices are included in the alternating graph.

Let $S^*$ consist of all vertices at odd layers in the alternating graph. Then by construction this is
a vertex cover of $G$. Moreover since vertices in the odd layers are all matched by $M^*$, $|S^*| = |M^*|$, so by the above lemma, the minimum vertex cover has the same size as the maximum matching.

For $G$ which is not connected, clearly the minimum vertex cover of $G$ has size equal to the sum
of the minimum vertex covers of the connected components of $G$, and similarly for the maximum
matching. By the above result for connected graphs, we have $|M^*| = |S^*|$ for any bipartite $G$. □

We next show that Konig’s Theorem implies Hall’s Theorem:
If $G$ does not have a perfect matching, then there is a vertex cover $T$ with $|T| < n$ by Konig’s
Theorem. Take $S = A \setminus T$, then $N(S) \subseteq T \cap B$ so $|N(S)| \leq |T \cap B| < n - |T \cap A| = |S|$. This is the
harder direction of Hall’s Theorem. □

2 Polyhedral Formulation of Perfect Matchings in Bipartite Graphs

Assume the graph has a perfect matching. Can we find the matching by solving a linear program-
ing relaxation of the problem?
Given a graph $G(V, E)$ and a matching $M$, let $x(M) \in \mathbb{R}^{|E|}$ be the indicator vector of matching $M$:

$$
 x_{ij}(M) = \begin{cases} 
 1 & e = \{i,j\} \in M \\
 0 & \text{otherwise}
\end{cases}.
$$

Consider the following polytope $P_{LP}$ consisting of $x$ such that

$$
\sum_{j \in B} x_{ij} = 1 \forall i \in A, \\
\sum_{i \in A} x_{ij} = 1 \forall j \in B, \\
x_{ij} \geq 0.
$$

Define the set $P_M$ as convex hull of $x(M)$ for all perfect matchings.

**Theorem 7** If $G$ is a bipartite graph then $P_{LP} = P_M$.

**Proof:** Showing $P_M \subseteq P_{LP}$ is easy. Since each vector $x(M)$ for a perfect matching $M$ satisfies $\sum_{j \in B} x_{ij} = \sum_{i \in A} x_{ij} = 1$. Any convex combination of these vectors thus also satisfies the matching condition. Thus, $P_M \subseteq P_{LP}$.

To show $P_{LP} \subseteq P_M$, we show all corner points of $P_{LP}$ are integral (hence they are indicator vectors of perfect matchings, which lie in $P_M$). Let $x$ be a corner point. If $x_{i_1i_2} \in (0,1)$ for some $\{i_1, i_2\} \in E$, since $\sum_j x_{ij} = 1$, there must be $i_3 \neq i_1$ such that $x_{i_2i_3} \in (0,1)$. We continue this process, since the graph is finite, there is some $k$ such that $i_k = i_j$ for $j < k - 1$. Then we have a cycle of edges with weight in $(0,1)$, which must be an even cycle as $G$ is bipartite.

Let $C$ be such an even cycle and $\epsilon$ be the minimum edge value in this cycle. Let $d = (\ldots, \epsilon, \ldots, -\epsilon, \ldots, \epsilon, \ldots, -\epsilon)$ where $\epsilon$ and $-\epsilon$ appear in the coordinates corresponding to the edges in $C$ and the sign alternates along the cycle, and $d$ is 0 elsewhere. One can see that $x + d$ and $x - d$ are both feasible solutions of $P_{LP}$ and $x$ can be written as a convex combination of $x + d$ and $x - d$. This contradicts the fact that $x$ is a corner point of $P_{LP}$.

Consider the following LP which searches for the maximum weight matching in $G$.

$$
\text{maximize} \quad \sum_{i \in A, j \in B} v_{ij} x_{ij} \\
\text{s.t.} \quad \sum_{j \in B} x_{ij} = 1 \forall i \in A, \\
\sum_{i \in A} x_{ij} = 1 \forall j \in B, \\
x_{ij} \geq 0.
$$

(1, 2, 3, 4)

Since $P_{LP} = P_M$, there is always a perfect matching that maximizes the utility $\sum_{i \in A, j \in B} v_{ij} x_{ij}$. 

We can write the dual of the above LP.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in A} e_i + \sum_{j \in B} p_j \\
\text{s.t.} & \quad e_i + p_j \geq v_{ij} \\
& \quad e_i, p_j \geq 0
\end{align*}
\] (5)

The complimentary slackness condition then gives: if \( x_{ij} > 0 \) then \( p_j + e_i = v_{ij} \), if \( p_j > 0 \) then \( \sum_i x_{ij} = 1 \), if \( e_i > 0 \) then \( \sum_j x_{ij} = 1 \). We can give the following interpretation to the dual solution. Consider \( B \) as a set of sellers and \( A \) as a set of buyers. Then \( p_j \) gives the prices of the object \( j \) sells (utility of \( j \)), and \( e_i \) gives the excess utility of the buyers, \( v_{ij} \) is the value of object \( j \) for buyer \( i \). Note that \( e_i = \max_j (v_{ij} - p_j) \). Thus, the primal LP gives a matching that maximizes total utility, and the dual LP gives prices that support the allocation by the primal LP.