1 Online Budgeted Matching

Let $L$ be the set of search queries, which arrive in an online fashion, and let $R$ be the set of advertisers. Advertiser $j$ has budget $B_j$ and bids $b_{ij}$ on search query $i$. Note that this bid is only observed at the time when query $i$ arrives.

We model the corresponding offline primal program:

$$\text{max} \sum_{i \in L, j \in R} x_{ij} b_{ij}$$

$$\text{s.t.} \sum_{j \in R} x_{ij} \leq 1 \quad \forall i \in L$$

$$\sum_{i \in L} x_{ij} b_{ij} \leq B_j \quad \forall j \in R$$

$$x_{ij} \geq 0 \quad \forall i \in L, j \in R$$

Note that the last constraint is a relaxation of the integer constraint $x_{ij} \in \{0, 1\}$ for all $i \in L, j \in R$. The associated dual program is

$$\text{min} \sum_{i \in L} \alpha_i + \sum_{j \in R} B_j \beta_j$$

$$\text{s.t.} \alpha_i + b_{ij} \beta_j \geq b_{ij} \quad \forall i \in L$$

$$\sum_{i \in L} x_{ij} b_{ij} \leq B_j \quad \forall j \in R$$

$$\alpha_i \geq 0, \beta_j \geq 0 \quad \forall i \in L, j \in R$$

1.1 GREEDY

First, consider the GREEDY algorithm: when query $i$ arrives, match it to the highest bidder $j^*$ among the advertisers whose budget is not exhausted. As a technical note, we will consider capped bids $\tilde{b}_{ij} = \min\{b_{ij}, B_j - y_j\}$ in our analysis, where $y_j = \sum_{i':i' \text{ arrives before } i} x_{i'j} b_{i'j}$ is the amount already spent by advertiser $j$ when query $i$ arrives.

Theorem 1 GREEDY is 1/2-competitive with respect to the offline linear program.

Proof: We prove via the primal-dual method. Before any query arrives, we set $\alpha_i \leftarrow 0, \beta_j \leftarrow 0$ for all $i \in L, j \in R$. Now, suppose query $i$ arrives and GREEDY matches it to advertiser $j^*$. 
We set $\alpha_i \leftarrow \frac{1}{2} \hat{b}_{ij}$, and $\beta_j \leftarrow \beta_j + \frac{\hat{b}_{ij}}{2B_j}$. Under this assignment scheme, the change in the primal objective value is $\hat{b}_{ij}$, and the change in the dual objective value is

$$\frac{1}{2} b_{ij} + B_j \left( \frac{\hat{b}_{ij}}{2B_j} \right) = \hat{b}_{ij};$$

that is, the change in objective value is equal for both the primal and dual linear programs. In addition to comparing the objective value changes, we must also show that this assignment scheme is dual-feasible. Fix some query $i \in L$ and some advertiser $j \in R$ (where $j$ is not necessarily matched to $i$), and observe that there are three cases to consider.

**Case 1:** If advertiser $j$’s budget is exhausted, then $\beta_j = \frac{1}{2}$, and therefore

$$\alpha_i + \beta_j b_{ij} \geq \frac{1}{2} b_{ij}.$$ 

Notice that the inequality follows from the fact that $\alpha_i \geq 0$ by virtue of the assignment scheme.

**Case 2:** If advertiser $j$’s budget is not exhausted and $\beta_j = b_{ij}$, then $\alpha_i \geq \frac{1}{2} b_{ij}$. Therefore,

$$\alpha_i + \beta_j b_{ij} \geq \frac{1}{2} b_{ij},$$

where the inequality similarly follows from the non-negativity of $\beta_j b_{ij}$.

**Case 3:** If advertiser $j$’s budget is almost exhausted so that $\beta_j = B_j - y_j < b_{ij}$, then $\alpha_i \geq \frac{1}{2} b_{ij}$ and $\beta_j = \frac{y_j}{2B_j} = \frac{B_j - b_{ij}}{2B_j} = \frac{1}{2} \left( 1 - \frac{b_{ij}}{B_j} \right)$. Therefore,

$$\alpha_i + \beta_j b_{ij} \geq \frac{1}{2} b_{ij} + \frac{1}{2} b_{ij} \left( 1 - \frac{\hat{b}_{ij}}{B_j} \right) = \frac{1}{2} b_{ij} + \frac{1}{2} b_{ij} \left( 1 - \frac{\hat{b}_{ij}}{B_j} \right) \geq \frac{1}{2} b_{ij},$$

where the inequality follows from the assumption that $b_{ij} \leq B_j$, making the second term on the righthand side non-negative.

Now, in order to make this assignment scheme dual feasible, let $\{\alpha_i, \beta_j\} = \{2\alpha_i, 2\beta_j\}$. Under this modified assignment scheme, when query $i$ is matched to advertiser $j^*$, the primal objective value increases by $b_{ij}^*$ while the dual objective values increases by $\frac{1}{2} b_{ij}^*$, from which it follows that GREEDY is $1/2$-competitive.

Note that this $1/2$ is tight. Suppose we have just two advertisers $j_1, j_2$, each with a budget of 1, and two search queries $i_1, i_2$. Also suppose advertiser $j_1$ would bid 1 on query $i_1$ and 1 on query $i_2$, whereas advertiser $j_2$ would bid $1 - \epsilon$ on query $i_1$, for some $0 < \epsilon < 1$, and 0 on query $i_2$. The optimal assignment is clearly to match query $i_1$ to advertiser $j_2$ and query $i_2$ to advertiser $j_1$, for a total value of $2 - \epsilon$. However, if query $i_1$ arrives first, GREEDY matches $i_1$ to advertiser $j_1$ (the higher bidder), so that when query $i_2$ arrives, advertiser $j_1$’s budget is exhausted. Therefore, the assignment prescribed by GREEDY algorithm results in a total value of 1.
1.2 BALANCE

We next briefly consider the BALANCE algorithm: when query $i$ arrives, match it to the advertiser $j^*$ that has spent the least so far. Notice, however, that this does not take into account the advertiser’s bid or total budget.

1.3 MSVV: Mehta, Saberi, Vazirani, Vazirani (FOCS ’05)

In [1], the authors propose an algorithm similar to GREEDY, with the difference being that when determining which advertiser to match a query to, they effectively reshape or scale each advertiser’s bid by the fraction of the advertiser’s remaining budget. More precisely, MSVV matches queries and advertisers according to the following: when query $i$ arrives, match it to the advertiser $j = \arg \max_j y_j \leq B_j, b_{ij} (1 - g(y_j/B_j))$, for some non-negative, non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ which we will fix later.

**Theorem 2** MSVV is $(1 - O(\epsilon))(1 - 1/e)$-competitive with respect to the offline linear program if $b_{ij}/B_j \leq \epsilon$ for all $i \in L, j \in R$.

**Proof:** We prove via the primal-dual method. We construct the dual variable assignment scheme so that although the dual is not feasible at the beginning it becomes feasible by the end, and also so that the change in the primal objective value is at least some constant $c$ times the change in the dual objective value,

$$\Delta \text{ primal} \geq c \cdot \Delta \text{ dual},$$

which implies MSVV is a $1/c$-competitive.

Before any query arrives, we set $\alpha_i \leftarrow 0, \beta_j \leftarrow 0$ for all $i \in L, j \in R$. Now, suppose query $i$ arrives and MSVV matches it to advertiser $j^*$. We set $\alpha_i \leftarrow b_{ij^*} (1 - g(y_{j^*}/B_{j^*}))$ and $\beta_{j^*} \leftarrow g(y_{j^*}/B_{j^*})$. Note that we only set $\alpha_i$ once (when query $i$ arrives) but we reset $\beta_{j^*}$ every time advertiser $j^*$ is matched to another query.

We must demonstrate that this assignment scheme is dual-feasible. Fix some query $i \in L$ and some advertiser $j \in R$ (where $j$ is not necessarily matched to $i$), and observe that there are two cases to consider.

**Case 1:** Query $i$ is matched to some advertiser $j'$, and therefore $\alpha_i = b_{ij'} (1 - g\left(\frac{y_{j'}}{B_{j'}}\right))$. If $y_j \leq B_j - b_{ij}$, then at the time query $i$ arrived, advertiser $j$ was a candidate for being matched with $i$, and therefore

$$\alpha_i + \beta_j b_{ij} = b_{ij'} \left(1 - g\left(\frac{y_{j'}}{B_{j'}}\right)\right) + b_{ij} \cdot g\left(\frac{y_j}{B_j}\right) \geq b_{ij} \left(1 - g\left(\frac{y_j}{B_j}\right)\right) + b_{ij} \cdot g\left(\frac{y_j}{B_j}\right) = b_{ij}.$$

Otherwise, if $y_j > B_j - b_{ij}$, then

$$g\left(\frac{y_j}{B_j}\right) \geq g\left(\frac{B_j - b_{ij}}{B_j}\right) \geq g(1 - \epsilon) \geq g(1) - \epsilon \cdot g'(1) = 1 - \epsilon \cdot g'(1)$$
where the first and second inequality follow from the fact that \( g \) is non-decreasing, the third inequality holds if we impose the condition that \( g \) is convex, and the fourth equality holds if we further assume that \( g(1) = 1 \). Then,
\[
\alpha_i + \beta_j b_{ij} \geq b_{ij} (\epsilon \cdot g'(1)) + b_{ij} (1 - \epsilon \cdot g'(1)) = b_{ij},
\]
which is feasible.

Case 2: Query \( i \) cannot be matched to any advertiser \( j' \) (including advertiser \( j \)), which means \( \alpha_i = 0 \) and \( b_{ij} > B_j - y_j \), or equivalently \( y_j > B_j - b_{ij} \). Therefore,
\[
\alpha_i + \beta_j b_{ij} = b_{ij} \cdot g \left( \frac{y_j}{B_j} \right) \\
\geq b_{ij} \cdot g \left( \frac{B_j - b_{ij}}{B_j} \right) \\
\geq b_{ij} \cdot g \left( 1 - \epsilon \right) \\
\geq b_{ij} \left( g(1) - \epsilon \cdot g'(1) \right) \\
= b_{ij} (1 - \epsilon)
\]
where the last equality holds if we additionally assume that \( g'(1) \leq 1 \). Notice that this is almost dual feasible.

Now, we compare the change in the primal and dual objective values when query \( i \) arrives. If query \( i \) is not matched, then the change in both objectives is zero. Otherwise, if query \( i \) is matched to advertiser \( j^* \), then the change in the primal objective value is \( b_{ij^*} \), and we can lower-bound the change in the dual objective as follows:
\[
\Delta \text{ dual} = b_{ij^*} \left( 1 - g \left( \frac{y_{j^*}}{B_{j^*}} \right) \right) + B_{j^*} \left( g \left( \frac{y_{j^*} + b_{ij^*}}{B_{j^*}} \right) - g \left( \frac{y_{j^*}}{B_{j^*}} \right) \right) \\
\geq b_{ij^*} \left( 1 - g \left( \frac{y_{j^*}}{B_{j^*}} \right) \right) + B_{j^*} \left( b_{ij^*} \frac{y_{j^*}}{B_{j^*}} \cdot g' \left( \frac{y_{j^*}}{B_{j^*}} \right) \right) \\
= b_{ij^*} \left( 1 - g \left( \frac{y_{j^*}}{B_{j^*}} \right) + g' \left( \frac{y_{j^*}}{B_{j^*}} \right) \right),
\]
where the first inequality follows from the convexity of \( g \). This means that \( c = 1 - g \left( \frac{y_{j^*}}{B_{j^*}} \right) + g' \left( \frac{y_{j^*}}{B_{j^*}} \right) \).

We must now define \( g \) so that all of the assumptions we have made on its functional form are satisfied and also so that \( c \) is minimized (in order to get the best approximation factor). We find \( g(x) = (e^x - 1)/(e - 1) \), which satisfies that \( g \) is a non-decreasing convex function for which \( g(0) \geq 0 \), \( g(1) = 1 \), and \( g'(1) \leq 1 \). Therefore, \( 1 - g(x) + g'(x) = e/(e - 1) \), from which it follows that \( \text{MSVV} \) is \( 1 - 1/e \)-competitive.

\[\Box\]

References