1. Denote the second eigenvalue of the Laplacian of $G(V,E)$ by $\lambda_2(G)$. Prove that for every $S \subseteq V$, $\lambda_2(G) \leq \lambda_2(G \setminus S) + |S|$. Use that to bound the second eigenvalue by vertex connectivity.

2. Show that for any symmetric matrix $X$ and any integer $k \geq 1$ the sum of the $k$ largest eigenvalues of $X$ is a convex function of $X$.

3. If $G$ has diameter $d$, then its adjacency matrix has at least $d + 1$ distinct eigenvalues.

4. Remember a spanning tree $T$ is $\alpha$-thin for graph $G(V,E)$ if and only if for every cut $(S, \bar{S})$

   $E_T(S, \bar{S}) \leq \alpha E_G(S, \bar{S}).$

   Prove that a $k$-dimensional hypercube has an $O(1/k)$-thin spanning tree. You can follow the following line of reasoning or find an original proof.

   - Suppose $G$ has a set of cycles $C_1, C_2, \ldots, C_k$ such that (i) each cycle has exactly one edge of $T$, and each edge of $T$ is an at least $\beta$ cycles. (ii) Each edge not in $T$ is in at most $\alpha$ cycles. Show that $T$ is $\alpha/\beta$-thin.
   - The next step to construct a connected thin subgraph. Decompose $H_{2k}$ into $2^k$ subcubes $H(x)$ for $0 \leq x < 2^k$ each uniquely determined by the first $k$ bits. Decompose each $H(x)$ into $k/2$ edge disjoint Hamiltonian paths and choose one from each decomposition in a way that their union is thin for $H_{2k}$.
   - Repeat the same for the last $k$ digits and show the union is connected.

5. Let $\alpha(G, \lambda)$ denote the matching polynomial of $G$. Prove that

   $\alpha(C_n, 2\lambda) = 2T_n(\lambda),$

   where $C_n$ is the cycle of length $n$ and $T_n$ is Chebyshev polynomial and

   $\alpha(K_{b,b}, \lambda) = (-1)^b L_b(\lambda^2),$

   where $K_{b,b}$ is the complete bipartite graph with $2b$ vertices and $L_b$ is the Laguerre polynomial.