MS&E 337 Spectral Graph Theory and Algorithmic Applications
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Homework #2

1. Denote the second eigenvalue of the Laplacian of $G(V,E)$ by $\lambda_2(G)$. Prove that for every $S \subseteq V$, $\lambda_2(G) \leq \lambda_2(G \setminus S) + |S|$. Use that to bound the second eigenvalue by vertex connectivity.

2. Show that for any symmetric matrix X and any integer $k \geq 1$ the sum of the k largest eigenvalues of X is a convex function of X.

3. if G has diameter d, then its adjacency matrix has at least $d+1$ distinct eigenvalues.

4. Remember a spanning tree T is α-thin for graph $G(V,E)$ if and only if for every cut $(S, \bar{S})$

$$E_T(S, \bar{S}) \leq \alpha E_G(S, \bar{S}).$$

Prove that a k-dimensional hypercube has an $O(1/k)$-thin spanning tree. You can follow the following line of reasoning or find an original proof.

   - Suppose G has a set of cycles $C_1, C_2, \cdots, C_k$ such that (i) each cycle has exactly one edge of T, and each edge of T is an at least β cycles. (ii) Each edge not in T is in at most α cycles. Show that T is α/β-thin.
   - The next step to construct a connected thin subgraph. Decompose H_{2k} into 2^k subcubes $H(x)$ for $0 \leq x < 2^k$ each uniquely determined by the first k bits. Decompose each $H(x)$ into $k/2$ edge disjoint Hamiltonian paths and choose one from each decomposition in a way that their union is thin for H_{2k}.
   - Repeat the same for the last k digits and show the union is connected.

5. Let $\alpha(G, \lambda)$ denote the matching polynomial of G. Prove that

$$\alpha(C_n, 2\lambda) = 2T_n(\lambda),$$

where C_n is the cycle of length n and T_n is Chebyshev polynomial and

$$\alpha(K_{b,b}, \lambda) = (-1)^b L_b(\lambda^2),$$

where $K_{b,b}$ is the complete bipartite graph with $2b$ vertices and L_b is the Laguerre polynomial.