MS&E 337: Spectral Graph Theory and Algorithmic Applications **Spring 2015**

Lecture 1: 4/2/2015

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

1.0.1 The Laplacian matrix and its spectrum

Let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The *adjacency matrix* A_G is defined as the $n \times n$ matrix where the non-diagonal entry a_{ij} is 1 iff $i \sim j$, i.e., there is an edge between vertex i and vertex j and 0 otherwise. Let $D(G)$ define an arbitrary orientation of the edges of G. The *(oriented) incidence matrix* B_D is an $n \times m$ matrix such that $q_{ij} = -1$ if the edge corresponding to column j leaves vertex i, 1 if it enters vertex i, and 0 otherwise. We may denote the adjacency matrix and the incidence matrix simply by A and B when it is clear from the context.

One can discover many properties of graphs by observing the incidence matrix of a graph. For example, consider the following proposition.

Proposition 1.1. *If G has c connected components, then* $\text{Rank}(B) = n - c$.

Proof. We show that the dimension of the null space of B is c. Let z denote a vector such that $z^T B = \mathbf{0}$. This implies that for every $i \sim j$, $z_i = z_j$. Therefore z takes the same value on all vertices of the same connected component. Hence, the dimension of the null space is c. □

The *Laplacian matrix* $L = BB^T$ is another representation of the graph that is quite useful. Observe that

$$l_{ij} = \begin{cases} degree(i) & i = j \\ -1 & i \sim j \\ 0 & \text{otherwise} \end{cases}$$

You can also write $L = D - A$ where D is the diagonal $n \times n$ matrix where d_{ii} equals the degree of i in G.

Proposition 1.2. *The Laplacian matrix L is positive semi-definite and singular.*

Proof. Let λ be an eigenvalue v its corresponding eigenvector:

$$\lambda = v^T L v = (v^T Q)(Q^T v) = (Q^T v)^T (Q^T v) \geq 0.$$

Furthermore, L is singular since the summation of entries in every column is zero. □

We can write the eigenvalues of L as

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$$

One can also derive the eigenvalues using the following quadratic form. For a vector x, we have

$$x^T L x = x^T Q Q^T x = \sum_{i \sim j} (x_i - x_j)^2$$

The variational characterization of eigenvalues gives a way of estimating eigenvalues as solutions of an optimization problem.

$$\begin{aligned}\lambda_k &= \min_{x_1,\ldots,x_k \text{ orthogonal}} \max_{x\neq 0} \left\{ \frac{x^\intercal A x}{x^\intercal x} : x \in \text{span}\{x_1,\ldots,x_k\} \right\} \qquad (1.1)\\ &= \max_{x_1,\ldots,x_{n-k+1} \text{ orthogonal}} \min_{x\neq 0} \left\{ \frac{x^\intercal A x}{x^\intercal x} : x \in \text{span}\{x_1,\ldots,x_{n-k+1}\} \right\}.\end{aligned}$$

The spectrum of the Laplacian incorporates a number of combinatorial properties of the graph. For example, it is easy to check that $\text{tr}(L) = \sum_{i=0}^{n-1} \lambda_i = 2m$. How about the product of eigenvalues? The answer is far more interesting:

Theorem 1.3. $\frac{1}{n}\prod_{i=1}^{n-1} \lambda_i =$ *the number of spanning trees of* G*.*

1.0.2 The matrix-tree theorem

Let L_{ii} denote the $(n-1)\times(n-1)$ matrix obtained by removing row and column i from L. The proof of the theorem directly follows from the following.

Theorem 1.4 (The matrix-tree theorem)**.** $\sum_i det(L_{ii}) =$ *the number of spanning trees of* G*.*

Proof. Let C denote the matrix obtained by removing row and column i from Q. Since $L = QQ^T$, we get that $L_{ii} = CC^T$. By Cauchy-Binet formula, we have

$$det(L_{ii}) = \sum_N det(N)\cdot det(N^T)$$

where N iterates over all $(n-1)\times(n-1)$ submatrices of C. In other words, selecting N corresponds to selecting $n-1$ edges. The following claim shows that $det(N)^2$ is an indicator variable for a choice of N which takes the value 1 if N corresponds to a spanning tree, and 0 otherwise. Thus, the right-hand summation of the above equation is a count of the total number of spanning trees.

Claim 1.5. $det(N) = 0$ *if the corresponding edges have a cycle,* ± 1 *otherwise.*

Proof. Suppose there is a cycle among the edges corresponding to N. The columns that form the cycle are not independent and therefore N does not have a full rank. Hence, $det(N) = 0$ if the edges corresponding to N have a cycle.

On the other hand, suppose N is acyclic. We re-label the vertices (rows) and the edges (columns) to form a lower triangular matrix from N. We start with $k = 1$. We pick an arbitrary leaf from the tree that is not the original vertex i. We re-label the vertex and the attached edge as k. We remove the leaf from the tree and we increase k. We repeat this process until all vertices (except the original vertex i) are labeled. Observe that by construction, the matrix is lower rectangular and the diagonal entries are either 1 or -1. Therefore the determinant of such a matrix is either $+1$ or -1. □

□

For any square matrix M, the *characteristic polynomial* is defined as $p_M(t) = det(t\mathbf{I} - M)$. It is easy to see that the roots of the characteristic polynomial are the eigenvalues of the matrix.

The second ingredient we need is the following lemma.

Lemma 1.6. $\prod_{i=1}^{n-1} \lambda_i = \sum_i det(L_{ii})$.

Proof. Consider L's characteristic polynomial

$$p_L(t) = det(t\mathbf{I} - L) = t(t - \lambda_1) \cdots (t - \lambda_{n-1}).$$

The coefficient of t in this polynomial is $(-1)^{(n-1)} \prod_{i=1}^{n-1} \lambda_i$. We prove the lemma by showing that this coefficient also equals $(-1)^{(n-1)} \sum_i det(L_{ii})$.

Indeed, we can prove a stronger property. The following inequality can be driven by standard algebraic techniques. For every two square matrices A and B, we have

$$det(A + B) = \sum_S det(A_S, B_{\overline{S}})$$

where S iterates over every non-empty subset of $\{1, \ldots, n\}$. Applying this formula on $det(t\mathbf{I} + (-L))$, we get $p_L(t)$ equals summation $det((t\mathbf{I})_S, (-L)_{\overline{S}})$. Observe that $det((t\mathbf{I})_S, (-L)_{\overline{S}}) = \alpha t^{|S|}$ for some constant α. Therefore the coefficient of t^k in $p_L(t)$ is

$$(-1)^{n-k} \sum det(\text{principal minors of size } n - k)$$

This implies that the coefficient of t is $(-1)^{(n-1)} \sum_i det(L_{ii})$, as desired. □