1.0.1 The Laplacian matrix and its spectrum

Let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The adjacency matrix $A_G$ is defined as the $n \times n$ matrix where the non-diagonal entry $a_{ij}$ is 1 iff $i \sim j$, i.e., there is an edge between vertex $i$ and vertex $j$ and 0 otherwise. Let $D(G)$ define an arbitrary orientation of the edges of $G$. The (oriented) incidence matrix $B_D$ is an $n \times m$ matrix such that $q_{ij} = -1$ if the edge corresponding to column $j$ leaves vertex $i$, 1 if it enters vertex $i$, and 0 otherwise. We may denote the adjacency matrix and the incidence matrix simply by $A$ and $B$ when it is clear from the context.

One can discover many properties of graphs by observing the incidence matrix of a graph. For example, consider the following proposition.

**Proposition 1.1.** If $G$ has $c$ connected components, then $\text{Rank}(B) = n - c$.

**Proof.** We show that the dimension of the null space of $B$ is $c$. Let $z$ denote a vector such that $z^T B = 0$. This implies that for every $i \sim j$, $z_i = z_j$. Therefore $z$ takes the same value on all vertices of the same connected component. Hence, the dimension of the null space is $c$. \hfill \Box

The Laplacian matrix $L = B B^T$ is another representation of the graph that is quite useful. Observe that

$$l_{ij} = \begin{cases} \text{degree}(i) & i = j \\ -1 & i \sim j \\ 0 & \text{otherwise} \end{cases}$$

You can also write $L = D - A$ where $D$ is the diagonal $n \times n$ matrix where $d_{ii}$ equals the degree of $i$ in $G$.

**Proposition 1.2.** The Laplacian matrix $L$ is positive semi-definite and singular.

**Proof.** Let $\lambda$ be an eigenvalue $v$ its corresponding eigenvector:

$$\lambda = v^T L v = (v^T Q)(Q^T v) = (Q^T v)^T (Q^T v) \geq 0.$$ 

Furthermore, $L$ is singular since the summation of entries in every column is zero. \hfill \Box

We can write the eigenvalues of $L$ as

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$$

One can also derive the eigenvalues using the following quadratic form. For a vector $x$, we have

$$x^T L x = x^T Q Q^T x = \sum_{i \sim j} (x_i - x_j)^2$$

1-1
The variational characterization of eigenvalues gives a way of estimating eigenvalues as solutions of an optimization problem.

$$\lambda_k = \min_{x_1, \ldots, x_k \text{ orthogonal}} \max_{x \neq 0} \left\{ \frac{x^T A x}{x^T x} : x \in \text{span}\{x_1, \ldots, x_k\} \right\}$$

(1.1)

The spectrum of the Laplacian incorporates a number of combinatorial properties of the graph. For example, it is easy to check that $\text{tr}(L) = \sum_{i=0}^{n-1} \lambda_i = 2m$. How about the product of eigenvalues? The answer is far more interesting:

**Theorem 1.3.** \( \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i = \text{the number of spanning trees of } G. \)

1.0.2 The matrix-tree theorem

Let $L_{ii}$ denote the $(n-1) \times (n-1)$ matrix obtained by removing row and column $i$ from $L$. The proof of the theorem directly follows from the following.

**Theorem 1.4 (The matrix-tree theorem).** \( \sum_i \det(L_{ii}) = \text{the number of spanning trees of } G. \)

**Proof.** Let $C$ denote the matrix obtained by removing row and column $i$ from $Q$. Since $L = QQ^T$, we get that $L_{ii} = CC^T$. By Cauchy-Binet formula, we have

$$\det(L_{ii}) = \sum_N \det(N) \cdot \det(N^T)$$

where $N$ iterates over all $(n-1) \times (n-1)$ submatrices of $C$. In other words, selecting $N$ corresponds to selecting $n-1$ edges. The following claim shows that $\det(N)^2$ is an indicator variable for a choice of $N$ which takes the value 1 if $N$ corresponds to a spanning tree, and 0 otherwise. Thus, the right-hand summation of the above equation is a count of the total number of spanning trees.

**Claim 1.5.** $\det(N) = 0$ if the corresponding edges have a cycle, $\pm 1$ otherwise.

**Proof.** Suppose there is a cycle among the edges corresponding to $N$. The columns that form the cycle are not independent and therefore $N$ does not have a full rank. Hence, $\det(N) = 0$ if the edges corresponding to $N$ have a cycle.

On the other hand, suppose $N$ is acyclic. We re-label the vertices (rows) and the edges (columns) to form a lower triangular matrix from $N$. We start with $k = 1$. We pick an arbitrary leaf from the tree that is not the original vertex $i$. We re-label the vertex and the attached edge as $k$. We remove the leaf from the tree and we increase $k$. We repeat this process until all vertices (except the original vertex $i$) are labeled. Observe that by construction, the matrix is lower rectangular and the diagonal entries are either 1 or $-1$. Therefore the determinant of such a matrix is either $+1$ or $-1$. \hfill $\square$

For any square matrix $M$, the **characteristic polynomial** is defined as $p_M(t) = \det(tI - M)$. It is easy to see that the roots of the characteristic polynomial are the eigenvalues of the matrix.

The second ingredient we need is the following lemma.
Lemma 1.6. $\prod_{i=1}^{n-1} \lambda_i = \sum_i det(L_{ii})$.

Proof. Consider $L$’s characteristic polynomial

$$p_L(t) = det(tI - L) = t(t - \lambda_1) \cdots (t - \lambda_{n-1}).$$

The coefficient of $t$ in this polynomial is $(-1)^{(n-1)} \prod_{i=1}^{n-1} \lambda_i$. We prove the lemma by showing that this coefficient also equals $(-1)^{(n-1)} \sum_i det(L_{ii})$.

Indeed, we can prove a stronger property. The following inequality can be driven by standard algebraic techniques. For every two square matrices $A$ and $B$, we have

$$det(A + B) = \sum_S det(A_S, B_{\overline{S}})$$

where $S$ iterates over every non-empty subset of $\{1, \ldots, n\}$. Applying this formula on $det(tI + (-L))$, we get $p_L(t)$ equals summation $det((tI)_S, (-L)_{\overline{S}})$. Observe that $det((tI)_S, (-L)_{\overline{S}}) = \alpha t^{|S|}$ for some constant $\alpha$. Therefore the coefficient of $t^k$ in $p_L(t)$ is

$$(-1)^{n-k} \sum det(\text{principal minors of size } n-k)$$

This implies that the coefficient of $t$ is $(-1)^{(n-1)} \sum_i det(L_{ii})$, as desired. \qed