Quantitative Finance, Vol. 7, No. 2, April 2007, 189–216

Stochastic programming for funding mortgage pools

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(*Received 1 January 2006; in final form 12 October 2006*)

We present how to use stochastic programming to best fund a pool of similar fixed-rate mortgages through issuing bonds, callable and non-callable, of various maturities. We discuss the estimation of expected net present value and (downside) risk for different funding instruments using Monte Carlo simulation techniques, and the optimization of the funding using single- and multi-stage stochastic programming. Using realistic data we computed efficient frontiers of expected net present value versus downside risk for the single- and the multi-stage model, and studied the underlying funding strategies. Constraining the duration and convexity of the mortgage pool and the funding portfolios to match at any decision point, we computed duration and convexity hedged funding strategies and compared them with those from the multi-stage stochastic programming model without duration and convexity constraints. The out-of-sample results for the different data assumptions demonstrate that multi-stage stochastic programming yields significantly larger net present values at the same or at a lower level of risk compared with single-stage optimization and with duration and convexity hedging. We found that the funding strategies obtained from the multi-stage model differed significantly from those from the single-stage model and were again significantly different to funding strategies obtained from duration and convexity hedging. Using multi-stage stochastic programming for determining the best funding of mortgage pools will lead, in the average, to significantly higher profits compared with using single-stage funding strategies, or using duration and convexity hedging. An efficient method for the out-of-sample evaluation of strategies obtained from multi-stage stochastic programming models is presented.

Keywords: Stochastic programming; Funding; Mortgage pools

1. Introduction

Historically, the business of conduits, like Freddie Mac, Fannie Mae or Ginnie Mae, has been to purchase mortgages from primary lenders, pool these mortgages into mortgage pools, and securitize some if not all of the pools by selling the resulting Participation Certificates (PCs) to Wall Street. Conduits keep a fixed markup on the interest for their profit and roll over most of the (interest rate and prepayment) risk to the PC buyers. Recently, a more active approach, with the potential for significantly higher profits, has become increasingly attractive: instead of securitizing, funding the purchase of mortgage pools by issuing debt. The conduit firm raises the money for the mortgage purchases through a suitable combination of long- and short-term debt. Thereby, the conduit assumes a higher level of risk due to interest rate changes and prepayment risk but gains higher expected revenues due to the larger spread between the interest on debt and mortgage rates compared with the fixed markup by securitizing the pool.

The problem faced by the conduits is an asset-liability management problem, where the assets are the mortgages bought from primary lenders and the liabilities are the bonds issued. Asset liability problems usually are faced by pension funds and insurance companies. Besides assets, pension plans need to consider retirement obligations, which may depend on uncertain economic and institutional variables, and insurance companies need to consider uncertain pay-out obligations due to unforseen and often catastrophic events. Asset liability models are

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Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online
http://www.tandf.co.uk/journals
DOI: 10.1080/14697680601065566

most useful when both asset returns and liability pay-outs are driven by common, e.g. economic, factors. Often, the underlying stochastic processes and decision models are multi-dimensional and require multiple state variables for their representation. Using stochastic dynamic programming, based on Bellman's (1957) dynamic programming principle, for solving such problems is therefore computationally difficult, well known as the 'curse of dimensionality'. If the number of state variables of the problem is small, stochastic dynamic programming can be applied efficiently. Infanger (2006) discusses a stochastic dynamic programming approach for determining optimal dynamic asset allocation strategies over an investment horizon with many re-balancing periods, where the value-to-go function is approximated via Monte Carlo sampling. The paper uses an in-sample/out-of-sample approach to avoid optimization bias.

Stochastic programming can take into account directly the joint stochastic processes of asset and liability cash flows. Traditional stochastic programming uses scenario trees to represent possible future events. The trees may be constructed by a variety of scenario-generation techniques. The emphasis is on keeping the resulting tree thin but representative of the event distribution and on arriving at a computationally tractable problem, where obtaining a good first-stage solution rather than obtaining an entire accurate policy is the goal. Early practical applications of stochastic programming for asset liability management are reported in Kusy and Ziemba (1986) for a bank and in Carino *et al.* (1994) for an insurance company. Ziemba (2003) gives a summary of the stochastic programming approach for asset liability and wealth management. Early applications of stochastic programming for asset allocation are discussed in Mulvey and Vladimirou (1992), formulating financial networks, and Golub *et al.* (1995). Examples of early applications of stochastic programming for dynamic fixed-income strategies are Zenios (1993), discussing the management of mortgage-backed securities, Hiller and Eckstein (1993), and Nielsen and Zenios (1996). Wallace and Ziemba (2005) present recent applications of stochastic programming, including financial applications. Frauendorfer and Schüerle (2005) discuss the re-financing of mortgages in Switzerland.

Monte Carlo sampling is an efficient approach for representing multi-dimensional distributions. An approach, referred to as decomposition and Monte Carlo sampling, uses Monte Carlo (importance) sampling within a decomposition for estimating Benders cut coefficients and right-hand sides. This approach has been developed by Dantzig and Glynn (1990) and Infanger (1992). The success of the sampling within the decomposition approach depends on the type of serial dependency of the stochastic parameter processes, determining whether or not cuts can be shared or adjusted between different scenarios of a stage. Infanger (1994) and Infanger and Morton (1996) show that, for serial correlation of stochastic parameters (in the form of autoregressive processes), unless the correlation is limited to the right-hand side of the (linear) program, cut sharing is at best difficult for more than three-stage problems.

Monte Carlo pre-sampling uses Monte Carlo sampling to generate a tree, much like the scenario-generation methods referred to above, and then employs a suitable method for solving the sampled (and thus approximate) problem. We use Monte Carlo pre-sampling for representing the mortgage funding problem, and combine optimization and simulation techniques to obtain an accurate and tractable model. We also provide an efficient way to independently evaluate the solution strategy from solving the multi-stage stochastic program to obtain a valid upper bound on the objective. The pre-sampling approach provides a general framework of modeling and solving stochastic processes with serial dependency and many state variables; however, it is limited in the number of decision stages. Assuming a reasonable sample size for representing a decision tree, problems with up to four decision stages are meaningfully tractable. Dempster and Thorlacius (1998) discuss the stochastic simulation of economic variables and related asset returns. A recent review of scenario-generation methods for stochastic programming is given by Di Domenica *et al.* (2006), discussing also simulation for stochastic programming scenario generation.

In this paper we present how multi-stage stochastic programming can be used for determining the best funding of a pool of similar fixed-rate mortgages through issuing bonds, callable and non-callable, of various maturities. We show that significant profits can be obtained using multi-stage stochastic programming compared with using a single-stage model formulation and compared with using duration and convexity hedging, strategies often used in traditional finance. For the comparison we use an implementation of Freddie Mac's interest rate model and prepayment function. We describe in section 2 the basic formulation of funding mortgage pools and discuss the estimation of expected net present value and risk for different funding instruments using Monte Carlo sampling techniques. In section 3 we discuss the single-stage model. In section 4 we present the multi-stage model. Section 5 discusses duration and convexity and delta and gamma hedging. In section 6 we discuss numerical results using practical data obtained from Freddie Mac. We compare the efficient frontiers from the single-stage and multi-stage models, discuss the different funding strategies and compare them with delta and gamma hedged strategies, and evaluate the different strategies using out-of-sample simulations. In particular, section 6.5 presents the details of the out-of-sample evaluation of the solution strategy obtained from solving a multi-stage stochastic program. Section 7 reports on the solution of very large models and gives model sizes and solution times. Finally, section 8 summarizes the results of the paper.

While not explicitly discussed in this paper, the problem of what fraction of the mortgage pool should be securitized, and what portion should be retained and funded through issuing debt can be addressed through a minor extension of the models presented. Funding decisions for a particular pool are not independent of all other pools already in the portfolio and those to be acquired in the future. The approach can of course be extended to address also the funding of a number of pools with different characteristics. While the paper focuses on funding a pool of fixed-rate mortgages, the framework applies analogously to funding pools of adjustable-rate mortgages.

2. Funding mortgage pools

2.1. *Interest rate term structure*

Well-known interest rate term structure models in the literature are Vasicek (1977), Cox *et al.* (1985), Ho and Lee (1986), and Hull and White (1990), based on one factor, and Longstaff and Schwarz (1992) based on two factors.

Observations of the distributions of future interest rates are obtained using an implementation of the interest rate model of Luytjes (1993) and its update according to the Freddie Mac Document. The model reflects a stochastic process based on equilibrium theory using random shocks for short rate, spread (between the short rate and the ten-year rate) and inflation.

We do not use the inflation part of the model and treat it as a two-factor model, where the short rate and the spread are used to define the yield curve. To generate a possible interest rate path we feed the model at each period with realizations of two standard normal random variables and obtain as output for each period a possible outcome of a yield curve of interest rates based on the particular realizations of the random shocks. Given a realization of the short rate and the spread, the new yield curve is constructed free of arbitrage for all calculated yield points.

We denote as $i_t(m)$, $t = 1, \ldots, T$, the random interest rate of a zero coupon bond of term m in period t.

2.2. *The cash flows of a mortgage pool*

We consider all payments of a pool of fixed-rate mortgages during its lifetime. Time periods t range from $t = 0, \ldots, T$, where T denotes the end of the horizon; e.g. $T = 360$ reflects a horizon of 30 years considering monthly payments. We let B_t be the balance of the principal of the pool at the end of period t. The principal capital B_0 is given to the homeowners at time period $t = 0$ and is regained through payments β_t and through prepayments α_t at periods $t = 1, \ldots, T$. The balance of the principal is updated periodically by

$$B_t = B_{t-1}(1 + \kappa_0) - \beta_t - \alpha_t, \quad t = 1, \ldots, T.$$

The rate κ_0 is the contracted interest rate of the fixed-rate mortgage at time $t = 0$. We define λ_t to be the payment factor at period $t = 1, \ldots, T$. The payment factor when multiplied by the mortgage balance yields the constant monthly payments necessary to pay off the loan over its remaining life, e.g.

$$\lambda_t = \kappa_0/(1 - (1 + \kappa_0)^{t-T-1});$$

thus,

$$\beta_t = \lambda_t B_{t-1}.$$

The payment factor λ_t depends on the interest rate κ_0. For fixed-rate mortgages the quantity κ_0, and thus the quantities λ_t, are known with certainty. However, prepayments α_t, at periods $t = 1, \ldots, T$, depend on future interest rates and are therefore random parameters.

Prepayment models or functions represent the relationship between interest rates and prepayments. See, for example, Kang and Zenios (1992) for a detailed discussion of prepayment models and factors driving prepayments.

In order to determine α_t we use an implementation of Freddie Mac's prepayment function according to Lekkas and Luytjes (1993). Denoting the prepayment rates obtained from the prepayment function as γ_t, $t = 1, \ldots, T$, we compute the prepayments α_t in period t as

$$\alpha_t = \gamma_t B_{t-1}.$$

2.3. *Funding through issuing debt*

We consider funding through issuing bonds, callable and non-callable, with various maturities. Let ℓ be a bond with maturity m_ℓ, $\ell \in L$, where L denotes the set of bonds under consideration. Let $f_{\ell t}^{\tau}$ be the payment factor for period t, corresponding to a bond ℓ issued at period τ, $\tau \le t \le \tau + m_\ell$:

$$f_{\ell t}^{\tau} \equiv \begin{cases} +1, & \text{if } t - \tau = 0, \\ -(i_\tau(m_\ell) + s_{\tau\ell}), & \text{if } 0 < t - \tau < m_\ell, \\ -(1 + i_\tau(m_\ell) + s_{\tau\ell}), & \text{if } t - \tau = m_\ell, \end{cases}$$

where $i_\tau(m_\ell)$ reflects the interest rate of a zero coupon bond with maturity m_ℓ, issued at period τ, and $s_{\tau\ell}$ denotes the spread between the zero coupon rate and the actual rate of bond ℓ issued at τ. The spread $s_{\tau\ell}$ includes the spread of bullet bonds over zero coupon bonds (referred to as agency spread) and the spread of callable bonds over bullet bonds (referred to as agency call spread), and is computed according to the model specification given in the Freddie Mac document (Luytjes 1996).

Let M_τ^ℓ denote the balance of a bond ℓ at the time τ it is issued. The finance payments resulting from bond ℓ are

$$d_t^\ell = f_{\ell t}^{\tau} M_\tau^\ell, \quad t = \tau, \ldots, \tau + m_\ell,$$

from the time of issue (τ) until the time it matures ($\tau + m_\ell$) or, if callable, it is called. We consider the balance of the bullet from the time of issue until the time of maturity as

$$M_t^\ell = M_\tau^\ell, \quad t = \tau, \ldots, \tau + m_\ell.$$

2.4. Leverage ratio

Regulations require that, at any time t, $t = 0, \ldots, T$, equity is set aside against debt in an amount such that the ratio of the difference of all assets minus all liabilities to all assets is greater than or equal to a given value μ. Let E_t be the balance of an equity (cash) account associated with the funding. The equity constraint requires that

$$\frac{B_t + E_t - M_t}{B_t + E_t} \geq \mu,$$

where the total asset balance is the sum of the mortgage balance and the equity balance, $B_t + E_t$, and $M_t = \sum_\ell M_t^\ell$ is the total liability balance.

At time periods $t = 0, \ldots, T$, given the mortgage balance B_t, and the liability balance M_t, we compute the equity balance that fulfills the leverage ratio constraint with equality as

$$E_t = \frac{M_t - B_t(1-\mu)}{1-\mu}, \quad t = 0, \ldots, T.$$

We assume that the equity account accrues interest according to the short rate i_t(short), the interest rate of a 3-month zero coupon bond. Thus, we have the following balance equation for the equity account:

$$E_t = E_{t-1}(1 + i_{t-1}(\text{short})) + e_{t-1},$$

where e_t are payments into the equity account (positive) or payments out of the equity account (negative). Using this equation we compute the payments e_t to and from the equity account necessary to maintain the equity balance E_t computed for holding the leverage ratio μ.

2.5. Simulation

Using the above specification we may perform a (Monte Carlo) simulation in order to obtain an observation of all cash flows resulting from the mortgage pool and from financing the pool through various bonds. In order to determine in advance how the funding is carried out, we need to specify certain decision rules defining what to do when a bond matures, when to call a callable bond, at what level to fund, and how to manage profits and losses. For the experiment we employed the following six rules.

(i) Initial funding is obtained at the level of the initial balance of the mortgage pool, $M_0 = B_0$.
(ii) Since at time $t = 0$, $M_0 = B_0$, it follows that $E_0 = [\mu/(1-\mu)]B_0$, an amount that we assume to be an endowed initial equity balance.
(iii) When a bond matures, refunding is carried out using short-term debt (non-callable 3-month bullet bond) until the end of the planning horizon, each time at the level of the balance of the mortgage pool.
(iv) Callable bonds are called according to the call rule specification in Freddie Mac's document (Luytjes 1996). Upon calling, refunding is carried out using short-term debt until the end of the planning horizon, each time at the level of the balance of the mortgage pool.
(v) The leverage ratio (ratio of the difference of all assets minus all liabilities to all assets) is $\mu = 0.025$.
(vi) At each time period t, after maintaining the leverage ratio, we consider a positive sum of all payments as profits and a negative sum as losses.

According to the decision rules, when funding a mortgage pool using a single bond ℓ, we assume at time $t = 0$ that $M_0^\ell = B_0$, i.e. that exactly the amount of the initial mortgage balance is funded using bond ℓ. After bond ℓ matures refunding takes place using another bond (according to the decision rules, short-term debt, say, bond $\hat{\ell}$), based on the interest rate and the level of the mortgage balance at the time it is issued. If the initial bond ℓ is callable, it may be called, and then funding carried out through another bond (say, short-term debt $\hat{\ell}$). Financing based on bond $\hat{\ell}$ is continued until the end of the planning horizon, i.e. until $T - \tau < m_{\hat{\ell}}$, and no more bond is issued. Given the type of bond being used for refunding, and given an appropriate calling rule, all finance payments for the initial funding using bond ℓ and the subsequent refunding using bond $\hat{\ell}$ can be determined. We denote the finance payments accruing from the initial funding based on bond ℓ and its consequent refunding based on bond $\hat{\ell}$ as

$$d_t^\ell, \quad t = 1, \ldots, T.$$

Once the funding and the corresponding liability balance M_t^ℓ is determined, the required equity balance $E_t = E_t^\ell$ and the payments $e_t = e_t^\ell$ are computed.

2.6. The net present value of the payment stream

Finally, we define as

$$P_t^\ell = \beta_t + \alpha_t + d_t^\ell - e_t^\ell, \quad t = 1, \ldots, T, \; P_0 = M_0 - B_0 = 0$$

the sum of all payments in period t, $t = 0, \ldots, T$, resulting from funding a pool of mortgages (initially) using bond ℓ.

Let I_t be the discount factor for period t, i.e.

$$I_t = \prod_{k=1}^{t} (1 + i_k(\text{short})), \quad t = 1, \ldots, T, \; I_0 = 1,$$

where we use the short rate at time t, i_t(short), for discounting. The net present value (NPV) of the payment stream is then calculated as

$$r_\ell = \sum_{t=0}^{T} \frac{P_t^\ell}{I_t}.$$

So far, we consider all quantities that depend on interest rates as random parameters. In particular, P_t^ℓ is a random parameter, since β_t, α_t, d_t^ℓ, and e_t^ℓ are random parameters depending on random interest rates. Therefore, the net present value r_ℓ is a random parameter as well. In order to simplify the notation we do not label any specific outcomes of the random parameters. A particular run of the interest rate model requires $2T$ random outcomes of unit normal random shocks. *We now label a particular path of the interest rates obtained from one run of the interest rate model and all corresponding quantities with* ω. In particular, we label a realization of the net present value based on a particular interest rate path as r_ℓ^ω.

2.7. Estimating the expected NPV of the payment stream

We use Monte Carlo sampling to estimate the expected value of the NPV of a payment stream. Under a crude Monte Carlo approach to the NPV estimation, we sample N paths $\omega \in S$, $N = |S|$, using different observations of the distributions of the $2T$ random parameters as input to the interest rate model, and we compute r_ℓ^ω for each $\omega \in S$. Then, an estimate for the expected net present value (NPV) of the cash flow stream based on initial funding using bond ℓ is

$$\bar{r}_\ell = \frac{1}{N}\sum_{\omega\in S} r_\ell^\omega.$$

We do not describe in this document how we use advanced variance reduction techniques (e.g. importance sampling) for the estimation of the expected net present value of a payment stream. We refer to Prindiville (1996) for how importance sampling could be applied.

2.8. The expected NPV of a funding mix

Using simulation (as described above) we compute the net present value of the payment stream r_ℓ^ω for each realization $\omega \in S$ and each possible initial funding $\ell \in L$. The net present value of a funding mix is given by the corresponding convex combination of the net present values of the components $\ell \in L$, i.e.

$$r^\omega = \sum_{\ell\in L} r_\ell^\omega x_\ell, \quad \sum_{\ell\in L} x_\ell = 1, \quad x_l \geq 0,$$

where x_ℓ are non-negative weights summing to one. The expected net present value of a funding mix,

$$\bar{r} = \frac{1}{N}\sum_{\omega\in S} r^\omega,$$

is also represented as the convex combination of the expected net present values of the components $\ell \in L$, i.e.

$$\bar{r} = \sum_{\ell\in L} \bar{r}_\ell x_\ell, \quad \sum_{\ell\in L} x_\ell = 1, \quad x_\ell \geq 0.$$

2.9. Risk of a funding mix

In order to measure risk, we use as an appropriate asymmetric penalty function the negative part of the deviation of the NPV of a funding portfolio from a pre-specified target u, i.e.

$$v^\omega = \left(\sum_{\ell\in L} r_\ell^\omega x_\ell - u\right)^-,$$

and consider risk as the expected value of v^ω, estimated as

$$\bar{v} = \frac{1}{N}\sum_{\omega\in S} v^\omega.$$

A detailed discussion of this particular risk measure is given in Infanger (1996). The efficient frontier with risk as the first lower partial moment is also referred to as the 'put–call efficient frontier'; see, for example, Dembo and Mausser (2000).

3. Single-stage stochastic programming

Having computed the NPVs r_ℓ^ω for all initial funding options $\ell \in L$ and for all paths $\omega \in S$, we optimize the funding mix with respect to expected returns and risk by solving the (stochastic) linear program

$$\begin{aligned}
\min \quad & \frac{1}{N}\sum v^\omega = \bar{v}, \\
\text{s.t.} \quad & \\
& \sum_\ell r_\ell^\omega x_\ell + v^\omega \geq u, \quad \omega \in S, \\
& \sum_\ell \bar{r}_\ell x_\ell \geq \rho, \\
& \sum_\ell x_\ell = 1, \\
& x_\ell \geq 0, \\
& v^\omega \geq 0.
\end{aligned}$$

The parameter ρ is a pre-specified value that the expected net present value of the portfolio should exceed or be equal to. Clearly, $\rho \leq \rho^{\max} = \max_\ell\{\bar{r}_\ell\}$. Using the model we trace out an efficient frontier starting with $\rho = \rho^{\max}$ and successively reducing ρ until $\rho = 0$, each time solving the linear program to obtain the portfolio with the minimum risk $\bar{v}$ corresponding to a given value of ρ.

The single-stage stochastic programming model optimizes funding strategies based on decision rules defined over the entire planning horizon of $T = 360$ periods, where the net present value of each funding strategy using initially bond ℓ and applying the decision rules is estimated using simulation.

A variant of the model arises by trading off expected NPV and risk in the objective, with λ denoting the risk-aversion coefficient:

$$\min \quad -\sum_\ell \bar{r}_\ell x_\ell + \lambda \frac{1}{N} \sum v^\omega,$$

$$\text{s.t.}$$

$$\sum_\ell r_\ell^\omega x_\ell + v^\omega \geq u, \quad \omega \in S,$$

$$\sum_\ell x_\ell = 1,$$

$$x_\ell \geq 0,$$

$$v^\omega \geq 0.$$

For a risk aversion of $\lambda = 0$, risk is not part of the objective and expected NPV is maximized. The efficient frontier can be traced out by increasing the risk aversion λ successively from zero to very large values, where the risk term in the objective entirely dominates.

This approach is very different to Markowitz's (1952) mean variance analysis in that the distribution of the NPV is represented through scenarios (obtained through simulations over a long time horizon, considering the application of decision rules) and a downside risk measure is used for representing risk.

4. Multi-stage stochastic programming

In the following we relax the application of decision rules at certain decision points within the planning horizon, and optimize the funding decisions at these points. This leads to a multi-stage stochastic programming formulation.

We partition the planning horizon $\langle 0, T \rangle$ into n sub-horizons $\langle T_1, T_2 \rangle, \langle T_2, T_3 \rangle, \ldots, \langle T_n, T_{n+1} \rangle$, where $T_1 = 0$, and $T_{n+1} = T$. For the experiment, we consider $n = 4$, and partition at $T_1 = 0$, $T_2 = 12$, $T_3 = 60$, and $T_4 = 360$. We label the decision points at time $t = T_1$ as stage 1, at time $t = T_2$ as stage 2, and at time $t = T_3$ as stage 3 decisions. Funding obtained at the decision stages is labeled as $\ell_1 \in L_1$, $\ell_2 \in L_2$, and $\ell_3 \in L_3$ according to the decision stages. At time $t = T_4$, at the end of the planning horizon, the (stage 4) decision involves merely evaluating the net present value of each end point for calculating the expected NPV and risk. In between the explicit decision points, at which funding is subject to optimization, we apply the decision rules defined above.

Instead of interest rate paths as used in the single-stage model, we now use an interest rate tree with nodes at each stage. We consider $|S_2|$ paths $\omega_2 \in S_2$ between $t = T_1$ and $t = T_2$; for each node $\omega_2 \in S_2$ we consider $|S_3|$ paths $\omega_3 \in S_3$ between $t = T_2$ and $t = T_3$; for each node $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$ we consider $|S_4|$ paths $\omega_4 \in S_4$ between $t = T_3$ and $t = T_4$. Thus, the tree has $|S_2 \times S_3 \times S_4|$ end points. We may denote $S = \{S_2 \times S_3 \times S_4\}$ and $\omega = (\omega_2, \omega_3, \omega_4)$. Thus a particular path through the tree is now labeled as $\omega = (\omega_2, \omega_3, \omega_4)$ using an index for each partition. Figure 1 presents the decision tree of the multi-stage model for only two paths for each period.

The simulation runs for each partition of the planning horizon are carried out in such a way that the dynamics of the interest rate process and the prepayment function are fully carried forward from one partition to the next. Since the interest rate model and the prepayment function include many lagged terms and require the storing of 64 state variables, the application of dynamic

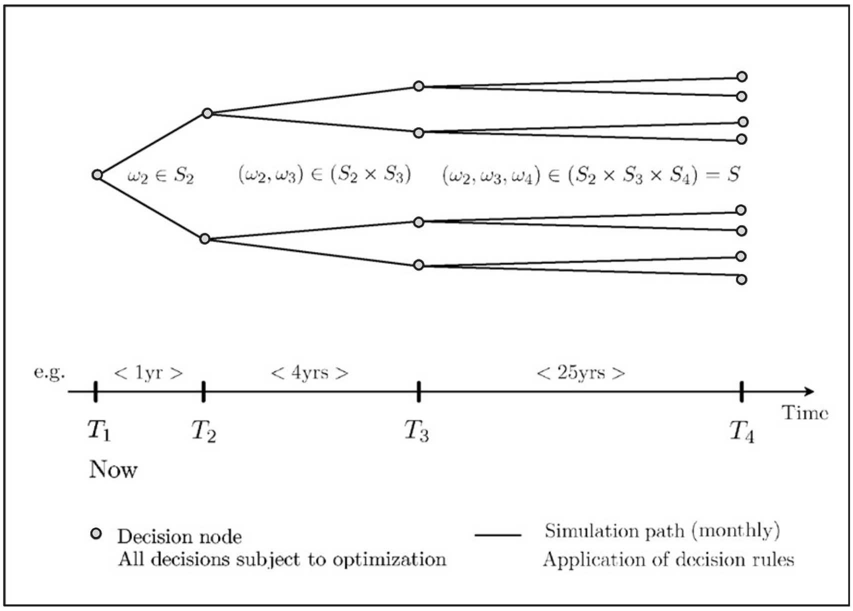

Figure 1. Multi-stage model setup, decision tree.

programming for solving the multi-stage program is not tractable.

Let $I_{\tau t}$ be the discount factor of period t, discounted to period τ, i.e.

$$I_{\tau t} = \prod_{k=\tau+1}^{t} (1 + i_k(\text{short})), \quad t > \tau, \; I_{\tau\tau} = 1,$$

where we use the short rate at time t, i_t(short), for discounting.

Let L_1 be the set of funding instruments available at time T_1. Funding obtained at time T_1 may mature or be called during the first partition (i.e. before or at time T_2), during the second partition (i.e. after time T_2 and before or at time T_3), or during the third partition (i.e. after time T_3 and before or at time T_4). We denote the set of funding instruments issued at time T_1 and matured or called during the first partition of the planning horizon as $L_{11}^{\omega_2}$, the set of funding instruments issued at time T_1 and matured or called during the second partition of the planning horizon as $L_{12}^{\omega_2\omega_3}$, and the set of funding instruments issued at time T_1 and matured or called during the third partition of the planning horizon as $L_{13}^{\omega_2\omega_3\omega_4}$. Clearly, $L_1 = L_{11}^{\omega_2} \cup L_{12}^{\omega_2\omega_3} \cup L_{13}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$. Similarly, we denote the set of funding instruments issued at time T_2 and matured or called during the second partition of the planning horizon as $L_{22}^{\omega_2\omega_3}$, and the set of funding instruments issued at time T_2 and matured or called during the third partition of the planning horizon as $L_{23}^{\omega_2\omega_3\omega_4}$. Clearly, $L_2 = L_{22}^{\omega_2\omega_3} \cup L_{23}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$. Finally, we denote the set of funding instruments issued at time T_3 and matured or called during the third partition of the planning horizon as L_{33}. Clearly, $L_3 = L_{33} = L_{33}^{\omega_2\omega_3\omega_4}$, for each $(\omega_2, \omega_3, \omega_4) \in \{S_2 \times S_3 \times S_4\}$.

For all funding instruments $\ell_1 \in L_1^{\omega_2}$ initiated at time $t=0$ that mature or are called during the first partition, we obtain the net present values

$$r_{\ell_1(11)}^{\omega_2} = \sum_{t=0}^{T_2} \frac{P_t^{\ell_1\omega_2}}{I_{0t}^{\omega_2}};$$

for all funding instruments $\ell_1 \in L_{12}^{\omega_2\omega_3}$ initiated at time $t=0$ that mature or are called during the second partition, we obtain the net present values

$$r_{\ell_1(12)}^{\omega_2,\omega_3} = \frac{1}{I_{0T_2}^{\omega_2}} \sum_{t=T_2+1}^{T_3} \frac{P_t^{\ell_1\omega_3}}{I_{T_2t}^{\omega_3}};$$

and all initial funding instruments $\ell_1 \in L_{13}^{\omega_2\omega_3\omega_4}$, initiated at time $t=0$ that mature or are called during the third partition, we obtain the net present values

$$r_{\ell_1(13)}^{\omega_2,\omega_3,\omega_4} = \frac{1}{I_{0T_2}^{\omega_2}} \frac{1}{I_{T_2T_3}^{\omega_3}} \sum_{t=T_3+1}^{T_4} \frac{P_t^{\ell_1\omega_4}}{I_{T_3t}^{\omega_4}}.$$

For all funding instruments $\ell_2 \in L_{22}^{\omega_2\omega_3}$, initiated at time $t = T_2$, that mature or are called during the second partition, we obtain the net present values

$$r_{\ell_2(22)}^{\omega_2,\omega_3} = \frac{1}{I_{0T_2}^{\omega_2}} \sum_{t=T_2+1}^{T_3} \frac{P_t^{\ell_2\omega_3}}{I_{T_2t}^{\omega_3}},$$

and for all funding instruments $\ell_2 \in L_{23}^{\omega_2\omega_3}$, initiated at time $t = T_2$, that mature or are called during the third partition, we obtain the net present values

$$r_{\ell_2(23)}^{\omega_2,\omega_3,\omega_4} = \frac{1}{I_{0T_2}^{\omega_2}} \frac{1}{I_{T_2T_3}^{\omega_3}} \sum_{t=T_3+1}^{T_4} \frac{P_t^{\ell_2\omega_4}}{I_{T_3t}^{\omega_4}}.$$

We obtain for all initial funding $\ell \in L_{33}$ initiated at time $t = T_3$ the net present values

$$r_{\ell_3(33)}^{\omega_2,\omega_3,\omega_4} = \frac{1}{I_{0T_2}^{\omega_2}} \frac{1}{I_{T_2T_3}^{\omega_3}} \sum_{t=T_3+1}^{T_4} \frac{P_t^{\ell_3\omega_4}}{I_{T_3t}^{\omega_4}}.$$

Let $N = |S|$. Let $R_{\ell_1}^{\omega} = r_{\ell_1(11)}^{\omega_2} + r_{\ell_1(12)}^{\omega_2\omega_3} + r_{\ell_1(13)}^{\omega_2\omega_3\omega_4}$, $R_{\ell_2}^{\omega} = r_{\ell_2(22)}^{\omega_2\omega_3} + r_{\ell_2(23)}^{\omega_2\omega_3\omega_4}$, and $R_{\ell_3}^{\omega} = r_{\ell_3(33)}^{\omega_2\omega_3\omega_4}$. Let x_{ℓ_1} be the amount of funding in instrument $\ell_1 \in L_1$ issued at time $t = T_1$, x_{ℓ_2} be the amount of funding in instrument $\ell_2 \in L_2$ issued at time $t = T_2$, and x_{ℓ_3} be the amount of funding in instrument $\ell_3 \in L_3$ issued at time $t = T_3$. Based on the computation of the net present values, we optimize the funding mix solving the multi-stage (stochastic) linear program:

$$\begin{aligned}
\min \quad E\, v^{\omega} &= \bar{v}, \\
\text{s.t.} & \\
\sum_{\ell_1 \in L_1} x_{\ell_1} &= 1, \\
-\sum_{\ell_1 \in L_{11}^{\omega_2}} x_{\ell_1} + \sum_{\ell_2 \in L_2} x_{\ell_2}^{\omega_2} &= 0, \\
-\sum_{\ell_1 \in L_{12}^{\omega_2\omega_3}} x_{\ell_1} - \sum_{\ell_2 \in L_{22}^{\omega_2\omega_3}} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} x_{\ell_3}^{\omega_2\omega_3} &= 0, \\
\sum_{\ell_1 \in L_1} R_{\ell_1}^{\omega} x_{\ell_1} + \sum_{\ell_2 \in L_2} R_{\ell_2}^{\omega} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} R_{\ell_3}^{\omega} x_{\ell_3}^{\omega_2\omega_3} - w^{\omega} &= 0, \\
v^{\omega} + w^{\omega} &\geq u, \\
E\, w^{\omega} &\geq \rho, \\
x_{\ell_1}, x_{\ell_2}^{\omega_2}, x_{\ell_3}^{\omega_2\omega_3}, v^{\omega} &\geq 0,
\end{aligned}$$

where $E\, w^{\omega} = (1/N) \sum w^{\omega}$ is the estimate of the expected net present value and $E\, v^{\omega} = (1/N) \sum v^{\omega}$ is the estimate of the risk. As in the single-stage model before, the parameter ρ is a pre-specified value for the expected net present value of the portfolio. Starting with $\rho = \rho^{\max}$, the maximum value of ρ that can be assumed without the linear program becoming infeasible, we trace out an efficient frontier by successively reducing ρ from $\rho = \rho^{\max}$ to $\rho = 0$ and computing for each level of ρ the corresponding value of risk $\bar{v}$ by solving the multi-stage stochastic linear program. The quantity $\rho^{\max}$, the maximum expected net present value without considering risk, can be obtained by solving the linear program

$$\max \quad E\, w^{\omega} = \rho^{\max},$$

$$\text{s.t.}$$

$$\sum_{\ell_1 \in L_1} x_{\ell_1} = 1,$$

$$-\sum_{\ell_1 \in L_{11}^{\omega_2}} x_{\ell_1} + \sum_{\ell_2 \in L_2} x_{\ell_2}^{\omega_2} = 0,$$

$$-\sum_{\ell_1 \in L_{12}^{\omega_2\omega_3}} x_{\ell_1} - \sum_{\ell_2 \in L_{22}^{\omega_2\omega_3}} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} x_{\ell_3}^{\omega_2\omega_3} = 0,$$

$$\sum_{\ell_1 \in L_1} R_{\ell_1}^{\omega} x_{\ell_1} + \sum_{\ell_2 \in L_2} R_{\ell_2}^{\omega} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} R_{\ell_3}^{\omega} x_{\ell_3}^{\omega_2\omega_3} - w^{\omega} = 0,$$

$$x_{\ell_1}, x_{\ell_2}^{\omega_2}, x_{\ell_3}^{\omega_2\omega_3} \geq 0.$$

Note that the model formulation presented above does not consider the calling of callable bonds as subject to optimization at the decision stages; rather the calling of callable bonds is handled through the calling rule as part of the simulation. Optimizing also the calling of callable bonds at the decision stages requires only a minor extension to the model formulation, but this is not discussed here.

A variant of the multi-stage model arises by trading off expected net present value and risk in the objective with λ as the risk-aversion coefficient:

$$\min \lambda E\, v^{\omega} - E\, w^{\omega},$$

$$\text{s.t.}$$

$$\sum_{\ell_1 \in L_1} x_{\ell_1} = 1,$$

$$-\sum_{\ell_1 \in L_{11}^{\omega_2}} x_{\ell_1} + \sum_{\ell_2 \in L_2} x_{\ell_2}^{\omega_2} = 0,$$

$$-\sum_{\ell_1 \in L_{12}^{\omega_2\omega_3}} x_{\ell_1} - \sum_{\ell_2 \in L_{22}^{\omega_2\omega_3}} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} x_{\ell_3}^{\omega_2\omega_3} = 0,$$

$$\sum_{\ell_1 \in L_1} R_{\ell_1}^{\omega} x_{\ell_1} + \sum_{\ell_2 \in L_2} R_{\ell_2}^{\omega} x_{\ell_2}^{\omega_2} + \sum_{\ell_3 \in L_3} R_{\ell_3}^{\omega} x_{\ell_3}^{\omega_2\omega_3} - w^{\omega} = 0,$$

$$v^{\omega} + w^{\omega} \geq u,$$

$$x_{\ell_1}, x_{\ell_2}^{\omega_2}, x_{\ell_3}^{\omega_2\omega_3}, v^{\omega} \geq 0.$$

5. Duration and convexity

Since the payments from a mortgage pool are not constant, indeed the prepayments depend on the interest rate term structure and its history since the inception of the pool, an important issue arises as to how the net present value (price) of the mortgage pool changes as a result of a small change in interest rates. The same issue arises for all funding instruments, namely, how bond prices change as a result of small changes in yield. This is especially interesting in the case of callable bonds. In order to calculate the changes in expected net present value due to changes in interest rates, one usually resorts to first- and second-order approximations, where the first-order (linear, or delta) approximation is called the duration and the second-order (quadratic, or gamma) approximation is called the convexity. While the duration and convexity of non-callable bonds could be calculated analytically, the duration and convexity of a mortgage pool and callable bonds can only be estimated through simulation. We use the terms effective duration and effective convexity to refer to magnitudes estimated through simulation.

5.1. *Effective duration and convexity*

Let

$$p = \sum_{t=0}^{T} \frac{P_t^{\text{pool}}}{I_t}$$

be the net present value of the payments from the mortgage pool, where $P_t^{\text{pool}} = \alpha_t + \beta_t$ and I_t is the discount factor using the short rate for discounting. We compute p^{ω}, for scenarios $\omega \in S$, using Monte Carlo simulation, and we calculate the expected net present value (price) of the payments of the mortgage pool as

$$\bar{p} = \frac{1}{N} \sum_{\omega \in S} p^{\omega}.$$

Note that the payments $P_t = P_t(i_k,\ k = 0, \ldots, t)$ depend on the interest rate term structure and its history up to period t, where i_t denotes the vector of interest rates for different maturities at time t. Writing explicitly the dependency,

$$\bar{p} = \bar{p}(i_t,\ t = 0, \ldots, T).$$

We now define

$$\bar{p}_+ = \bar{p}(i_t + \Delta,\ t = 0, \ldots, T)$$

as the net present value of the payments of the mortgage pool for an upward shift of all interest rates by Δ%, and

$$\bar{p}_- = \bar{p}(i_t - \Delta,\ t = 0, \ldots, T)$$

as the net present value of the payments of the mortgage pool for a downward shift of all interest rates by Δ%, where Δ is a shift of, say, one percentage point in the entire term structure at all periods $t = 1, \ldots, T$.

Using the three points $\bar{p}_-, \bar{p}, \bar{p}_+$, and the corresponding interest rate shifts $-\Delta$, 0, $+\Delta$, we compute the effective duration of the mortgage pool, as

$$\text{dur} = \frac{\bar{p}_- - \bar{p}_+}{2\Delta\bar{p}},$$

and the effective convexity of the mortgage pool,

$$\text{con} = \frac{\bar{p}_- + \bar{p}_+ - 2\bar{p}}{100\Delta^2\bar{p}}.$$

The quantities of the effective duration and effective convexity represent a local first-order (duration) and second-order (duration and convexity) Taylor approximation of the net present value of the payments of the mortgage pool as a function of interest. The approximation considers the effects on a constant shift of the entire yield curve across all points t, $t = 1, \ldots, T$. The number 100 in the denominator of the convexity simply scales the resulting numbers. The way it is computed, we expect a positive value for the duration, meaning that decreasing interest rates result in a larger expected net present value and increasing interest rates result in a smaller expected net present value. We also expect a negative value for the convexity, meaning that the function of price versus yield is locally concave.

In an analogous fashion we compute the duration dur_ℓ and the convexity con_ℓ for all funding instruments ℓ. Let

$$p_\ell = \sum_{t=0}^{m_\ell} \frac{d_t^\ell}{I_t}$$

be the net present value of the payments of the bond ℓ, where d_t^ℓ represents the payments of bond ℓ until maturity or until it is called. We compute p_ℓ^ω using Monte Carlo simulation over $\omega \in S$, and we calculate the expected net present value (price) of the payments for the bond ℓ as

$$\bar{p}_\ell = \frac{1}{N} \sum_{\omega \in S} p_\ell^\omega.$$

We calculate

$$\bar{p}_{\ell+} = \bar{p}_\ell(i_t + \Delta,\ t = 0, \ldots, T),$$

and

$$\bar{p}_{\ell-} = \bar{p}_\ell(i_t - \Delta,\ t = 0, \ldots, T),$$

the expected net present values for an upwards and downwards shift of interest rates, respectively. Analogously to the mortgage pool, we obtain the duration of bond ℓ as

$$\mathrm{dur}_\ell = \frac{\bar{p}_{\ell-} - \bar{p}_{\ell+}}{2\Delta \bar{p}_\ell},$$

and its convexity as

$$\mathrm{con}_\ell = \frac{\bar{p}_{\ell-} + \bar{p}_{\ell+} - 2\bar{p}_\ell}{100\Delta^2 \bar{p}_\ell}.$$

Since in the case of non-callable bonds the payments d_t^ℓ are fixed, changes in expected net present value due to changes in interest rates are influenced by the discount factor only. For non-callable bonds we expect a positive value for duration, meaning that increasing interest rates imply a smaller bond value and decreasing interest rates imply a larger bond value. We expect a positive value for convexity, meaning that the function of expected net present value versus interest rates is locally convex. In the case of callable bonds the behavior of the function of expected net present value versus interest rates is influenced not only by the discount rate but also by the calling rule. If interest rates decrease the bond may be called and the principal returned. The behavior of callable bonds is similar to that of mortgage pools in that we expect a positive value for duration and a negative value for convexity.

5.2. *Traditional finance: matching duration and convexity*

Applying methods of traditional finance, one would hedge interest rate risk by constructing a portfolio with a duration and a convexity of close to zero, respectively, thus achieving that the portfolio would exhibit no change in expected net present value (price) due to a small shift in the entire yield curve. Duration and convexity matching is also referred to as immunization (see, for example, Luenberger (1998) or as delta and gamma hedging.

In the situation of funding mortgage pools, hedging is carried out in such a way that a change in the price of the mortgage pool is closely matched by the negative change in the price of the funding portfolio, such that the change of the total portfolio (mortgage pool and funding) is close to zero. Thus the duration of the total portfolio is close to zero. In addition, the convexity of the mortgage pool is matched by the (negative) convexity of the funding portfolio, such that the convexity of the total portfolio (mortgage pool and funding) is close to zero. We write the corresponding duration and convexity hedging model as

$$\begin{aligned} \max \sum_\ell r_\ell x_\ell & \\ \sum_\ell \mathrm{dur}_\ell x_\ell - \mathrm{dg} &= \mathrm{dur}, \\ \sum_\ell \mathrm{con}_\ell x_\ell - \mathrm{cg} &= \mathrm{con}, \\ \sum_\ell x_\ell &= 1, \\ x_\ell &\geq 0, \end{aligned}$$

and

$$-\mathrm{dg}^{\max} \leq \mathrm{dg} \leq \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \leq \mathrm{cg} \leq \mathrm{cg}^{\max},$$

and expected net present value is maximized in the objective. The variable dg accounts for the duration gap, cg accounts for the convexity gap, $\mathrm{dg}^{\max}$ represents a predefined upper bound on the absolute value of the duration gap, and $\mathrm{cg}^{\max}$ a predefined upper bound on the absolute value of the convexity gap. Usually, when a duration and convexity hedged strategy is implemented, the model needs to be revised over time as the mortgage pool and the yield curve changes. In practise, updating the funding portfolio may be done on a daily or monthly basis to reflect changes in the mortgage pool due to prepayments and interest rate variations.

The duration and convexity hedging model is a deterministic model, uncertainty is considered as a shift of the entire yield curve, and hedged to the extent of the effect of the remaining duration and convexity gap.

5.3. Duration and convexity in the single-stage model

In the single-stage case we add the duration and convexity constraints

$$\sum_\ell \mathrm{dur}_\ell x_\ell - \mathrm{dg} = \mathrm{dur},$$
$$\sum_\ell \mathrm{con}_\ell x_\ell - \mathrm{cg} = \mathrm{con},$$

and

$$-\mathrm{dg}^{\max} \le \mathrm{dg} \le \mathrm{dg}^{\max}, \quad -\mathrm{cg}^{\max} \le \mathrm{cg} \le \mathrm{cg}^{\max}$$

to the single-stage linear program using the formulation in which expected net present value and risk are traded off in the objective. Setting the risk aversion λ to zero, only the expected net present value is considered in the objective, and the resulting single-stage stochastic program is identical to the duration and convexity hedging formulation from traditional finance as discussed in the previous section. This formulation allows one to constrain the absolute value of the duration and convexity gap to any specified level, to the extent that the single-stage stochastic program remains feasible. By varying the duration and convexity gap, we may study the effect of the resulting funding strategy on expected net present value and risk.

5.4. Duration and convexity in the multi-stage model

In the multi-stage model we wish to constrain the duration and convexity gap not only in the first stage, but at any decision stage and in any scenario. Thus, in the four-stage model discussed above we have one pair of constraint for the first stage, $|S_2|$ pairs of constraints in the second stage, and $|S_2 \times S_3|$ pairs of constraints in the third stage. Accordingly, we need to compute the duration and the convexity for the mortgage pool and all funding instruments at any decision point in all stages from one to three.

In order to simplify the presentation, we omit the scenario indices $\omega_2 \in S_2$ and $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$. Let dur_t and con_t be the duration and convexity of the mortgage pool in each stage t, and dur_{ℓ_t} and con_{ℓ_t} be the duration and convexity of the funding instruments issued at each period t.

Let dur_t^F and con_t^F be symbols used to conveniently present the duration and convexity of the funding portfolio at each period t. In the first stage, $\mathrm{dur}_1^F = \sum_{\ell_1 \in L_1} \mathrm{dur}_{\ell_1} x_{\ell_1}$. In the second stage, $\mathrm{dur}_2^F = \sum_{\ell_2 \in L_2} \mathrm{dur}_{\ell_2} x_{\ell_2} + \sum_{\ell_1 \in L_{12}} \mathrm{dur}_{\ell_1(12)} x_{\ell_1}$, where $\mathrm{dur}_{\ell_1(12)}$ represents the duration of the funding instruments from the first stage that are still available in the second stage. In the third stage, $\mathrm{dur}_3^F = \sum_{\ell_3 \in L_3} \mathrm{dur}_{\ell_3} x_{\ell_3} + \sum_{\ell_2 \in L_{23}} \mathrm{dur}_{\ell_2(23)} x_{\ell_2} + \sum_{\ell_1 \in L_{13}} \mathrm{dur}_{\ell_1(13)} x_{\ell_1}$, where $\mathrm{dur}_{\ell_2(23)}$ represents the duration of the funding instruments issued in the second stage still available in the third stage, and $\mathrm{dur}_{\ell_1(13)}$ represents the duration of the funding instruments from the first stage still available in the third stage.

Analogously, in the first stage, $\mathrm{con}_1^F = \sum_{\ell_1 \in L_1} \mathrm{con}_{\ell_1} x_{\ell_1}$. In the second stage, $\mathrm{con}_2^F = \sum_{\ell_2 \in L_2} \mathrm{con}_{\ell_2} x_{\ell_2} + \sum_{\ell_1 \in L_{12}} \mathrm{con}_{\ell_1(12)} x_{\ell_1}$, where $\mathrm{con}_{\ell_1(12)}$ is the convexity of the funding instruments from the first stage still available in the second stage. In the third stage, $\mathrm{con}_3^F = \sum_{\ell_3 \in L_3} \mathrm{con}_{\ell_3} x_{\ell_3} + \sum_{\ell_2 \in L_{23}} \mathrm{con}_{\ell_2(23)} x_{\ell_2} + \sum_{\ell_1 \in L_{13}} \mathrm{con}_{\ell_1(13)} x_{\ell_1}$, where $\mathrm{con}_{\ell_2(23)}$ is the convexity of the funding instruments issued in the second stage still available in the third stage, and $\mathrm{con}_{\ell_1(13)}$ is the convexity of the funding instruments from the first stage still available in the third stage.

Thus, in the multi-stage case, we add in each stage $t = 1, \ldots, 3$ and in each scenario $\omega_2 \in S_2$ and $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$ the constraints

$$\mathrm{dur}_t^F - \mathrm{dg}_t = \mathrm{dur}_t,$$
$$\mathrm{con}_t^F - \mathrm{cg}_t = \mathrm{con}_t,$$

and

$$-\mathrm{dg}_t^{\max} \le \mathrm{dg}_t \le \mathrm{dg}_t^{\max}, \quad -\mathrm{cg}_t^{\max} \le \mathrm{cg}_t \le \mathrm{cg}_t^{\max}.$$

The variables $\mathrm{dg}_t, \mathrm{cg}_t$ account for the duration and convexity gap, respectively, in each decisions stage and in each of the scenarios $\omega_2 \in S_2$ and $(\omega_2, \omega_3) \in \{S_2 \times S_3\}$. At each decision node the absolute values of the duration and convexity gap are constrained by $\mathrm{dg}_t^{\max}$ and $\mathrm{cg}_t^{\max}$, respectively. This formulation allows one to constrain the duration and convexity gap to any specified level, even at different levels at each stage, to the extent that the multi-stage stochastic linear program remains feasible.

Using the formulation of the multi-stage model, in which expected net present value and risk are traded off in the objective, and setting the risk aversion coefficient λ to zero, we obtain a model in which the duration and convexity are constrained at each node of the scenario tree and expected net present value is maximized. Since the scenario tree represents simulations of possible events of the future, the model results in a duration and convexity hedged funding strategy, where duration and convexity are constrained at the decision points of the scenario tree, but not constrained at points in between, where the funding portfolios are carried forward using the decision rules. We use this as an approximation for other duration and convexity hedged strategies, in which duration and convexity are matched, for example, at every month during the planning horizon. One could, in addition, apply funding rules in the simulation that result in duration and convexity hedged portfolios at every month, but such rules have not been applied in the computations used for this paper.

We are now in the position to compare the results from the multi-stage stochastic model trading off expected net present value and downside risk with the deterministic duration and convexity hedged model on the same scenario tree. The comparison looks at expected net present value and risk (represented by various measures), as well as underlying funding strategies.

6. Computational results

6.1. Data assumptions

For the experiment we used three data sets, based on different initial yield curves, labeled 'Normal', 'Flat', and 'Steep'. The data represent assumptions about the initial yield curve, the parameters of the interest rate model, and the prepayment function, assumptions about the funding instruments, assumptions about refinancing and the calling rule, and the planning horizon and its partitioning.

Table 1 presents the initial yield curve (corresponding to a zero coupon bond) for each data set. For each data set the mortgage contract rate is assumed to be one percentage point above the 10-year rate (labeled 'y10').

For the experiment we consider 16 different funding instruments. Table 2 presents the maturity, the time after which the instrument may be called, and the initial spread over the corresponding zero coupon bond (of the same maturity) for each instrument and for each of the data sets. For example, 'y03nc1' refers to a callable bond with a maturity of 3 years (36 months) and callable after 1 year (12 months). Corresponding to the data set 'Normal', it could be issued initially (at time $t=0$) at a rate of $6.12\% + 0.36\% = 6.48\%$, where 6.12% is the interest rate from table 1 and 0.36% is the spread from table 2.

We computed the results for a pool of \$100M. As the target for risk, u, we used the maximum expected net present value obtained using single-stage optimization, i.e. we considered risk as the expected net present value below this target, defined for each data set. In particular, the target for risk equals $u = 10.5$M for the 'Normal' data set, $u = 11.3$M for the 'Flat' data set, and $u = 18.6$M for the 'Steep' data set.

We first used single-stage optimization using $N=300$ interest rate paths. These results are not presented here. Then, in order to more accurately compare single-stage and multi-stage optimization, we used a tree with $N=4000$ paths, where the sample sizes in each stage are $|S_2| = 10$, $|S_3| = 20$ and $|S_4| = 20$. For this tree the multi-stage linear program has 8213 rows, 11 377 columns and 218 512 non-zero elements. The program can easily be solved on a modern personal computer in a very small amount of (elapsed) time. Also, the simulation runs to obtain the coefficients for the linear program can easily be carried out on a modern personal computer.

Table 1. Initial yield curves.

Label	Maturity (months)	Interest rate (%)		
		Normal	Flat	Steep
m03	3	5.18	5.88	2.97
m06	6	5.31	6.38	3.16
y01	12	5.55	6.76	3.36
y02	24	5.97	7.06	4.18
y03	36	6.12	7.36	4.58
y05	60	6.33	7.56	5.56
y07	84	6.43	7.59	5.97
y10	120	6.59	7.64	6.36
y30	360	6.87	7.76	7.20

6.2. Results for the single-stage model

As a base case for the experiment we computed the efficient frontier for each of the data sets using the single-stage model. Figure 2 presents the result for the 'Normal' data set in comparison with the efficient frontiers obtained from the multi-stage model. (The single-stage results obtained from optimizing on the tree closely resemble the efficient frontiers obtained from optimizing on 300 interest rate paths.) The efficient frontiers for the 'Flat' and 'Steep' data sets are similar in shape to that of the 'Normal' data set. While 'Normal' and 'Flat' have the typical shape one would expect, i.e. steep at low levels of risk and bending more flat with increasing risk, it is interesting to note that the efficient frontier for data set 'Steep' is very steep at all levels of risk.

As a base case we look at the optimal funding strategy at the point of 95% of the maximum expected net present value. In the graphs of the efficient frontiers this is the second point down from the point of the maximum expected net present value point.

The funding strategies for all three data sets are presented in table 3. The label 'j1' in front of the funding acronyms means that the instrument is issued in stage 1, and the label 'j2' refers to the instrument's issuance in stage 2. For example, a funding instrument called 'j1y07n' is a non-callable bond with a maturity of 7 years issued at stage 1. While the single-stage model naturally does not have a second stage, variables with

Table 2. Spreads for different funding instruments.

Label	Maturity (months)	Callable after (months)	Spread (%)		
			Normal	Flat	Steep
m03n	3		0.17	0.15	0.19
m06n	6		0.13	0.10	0.13
y01n	12		0.02	0.08	0.08
y02n	24		0.00	0.11	0.1
y03n	36		0.04	0.18	0.12
y03nc1	36	12	0.36	0.61	0.22
y05n	60		0.08	0.21	0.13
y05nc1	60	12	0.65	0.84	0.22
y05nc3	60	36	0.32	0.41	0.18
y07n	84		0.17	0.22	0.15
y07nc1	84	12	0.90	0.95	0.45
y07nc3	84	36	0.60	0.73	0.35
y10n	120		0.22	0.29	0.22
y10nc1	120	12	1.10	1.28	0.57
y10nc3	120	36	0.87	0.94	0.48
y30n	360		0.30	0.32	0.27

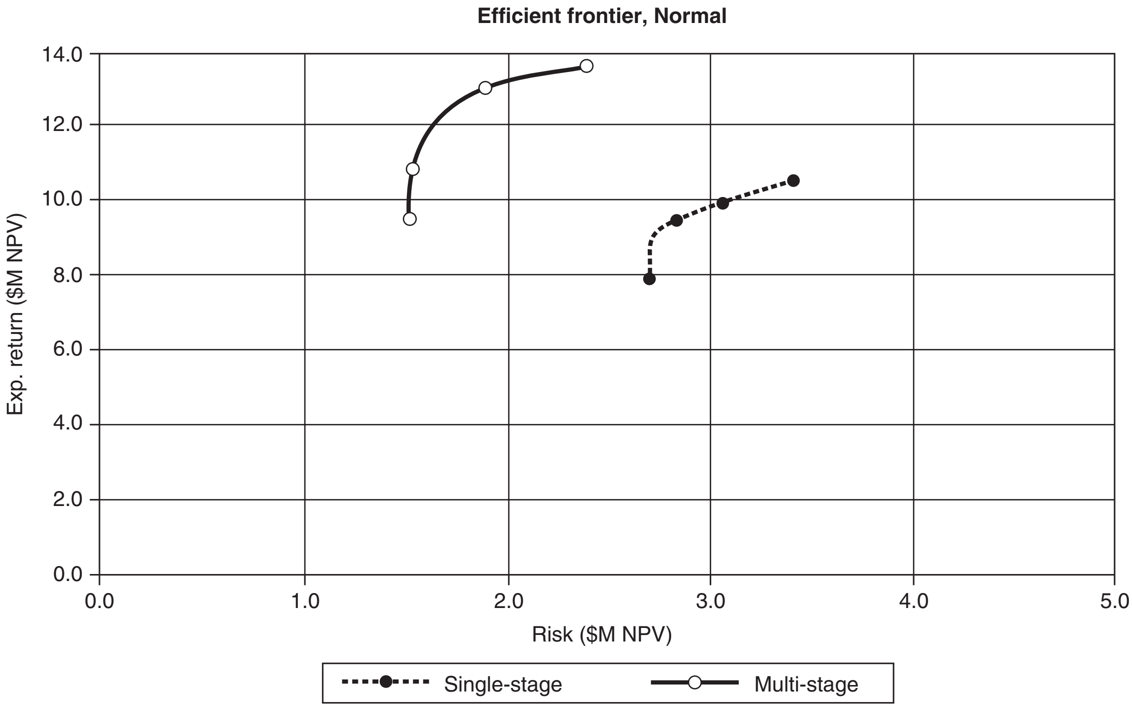

Figure 2. Efficient frontier for the Normal data set, single- versus multi-stage.

the prefix 'j2' for the second stage represent the amount of each instrument to be held at stage 2 according to the decision rules. This is to facilitate easy comparisons between the single-stage and multi-stage funding strategies.

In the 'Normal' case, the optimal initial funding mix consists of 85.3% six-month non-callable debt and 14.7% seven-year non-callable debt. The risk level of this strategy is 3.1M NPV. In the 'Flat' case, the optimal initial funding mix consists of 91.1% three-month non-callable debt and 8.9% seven-year non-callable debt. The risk level of this strategy is 3.1M NPV. In the 'Steep' case, the initial funding mix consists of 100.0% three-month non-callable debt (short-term debt). The risk level of this strategy is 2.6M NPV.

6.3. *Results for the multi-stage model*

Figure 2 presents the efficient frontier for the 'Normal' data set. In addition to the multi-stage efficient frontier, the graph also contains the corresponding single-stage efficient frontier for better comparison. The results for the 'Flat' and 'Steep' data sets are very similar in shape to that of the 'Normal' data set and therefore are not presented graphically. The results show substantial differences in the risk and expected net present value profile of multi-stage versus single-stage funding strategies. For any of the three data sets, the efficient frontier obtained from the multi-stage model is significantly north-west of that from the single-stage model, i.e. multi-stage optimization yields a larger expected net present value at the same or smaller level of risk.

Table 3. Funding strategies from the single-stage model, case 95% of maximum expected return, for all three data sets.

Model	Normal		Flat		Steep
Stage 1	Allocation				
	j1m06n	j1y07n	j1m03n	j1y07n	j1m03n
	0.853	0.147	0.911	0.089	1.000
Stage 2	Allocation				
Scenario	j2m03n		j2m03n		j2m03n
1	0.853		0.911		1.000
2	0.853		0.911		1.000
3	0.853		0.911		1.000
4	0.853		0.911		1.000
5	0.853		0.911		1.000
6	0.853		0.911		1.000
7	0.853		0.911		1.000
8	0.853		0.911		1.000
9	0.853		0.911		1.000
10	0.853		0.911		1.000

In all three data cases, we cannot compare the expected net present values from the single-stage and multi-stage model at the same level of risk, because the risk at the minimum risk point of the single-stage model is larger than the risk at the maximum risk point of the multi-stage model. In the 'Normal' case, the minimum risk of the single-stage curve is about 2.7M NPV. Since the efficient frontier is very steep at low levels of risk, we use the point of 85% of the maximum risk as the 'lowest risk' point, even if the risk could be further decreased by an insignificant amount. The maximum expected net present value point of the multi-stage curve has a risk of

Table 4. Funding strategy from the multi-stage model, case 95% of maximum expected return, Normal data set.

Stage 1	Allocation							
	j1m06n	j1y05nc1						
	0.785	0.215						
Stage 2	**Allocation**							
Scenario	**j2m03n**	**j2y03n**	**j2y03nc1**	**j2y05nc1**	**j2y10n**	**j2y10nc1**	**j2y10nc3**	**j2y30n**
1								1.000
2						0.597	0.188	
3		0.125	0.420		0.240			
4	0.785							
5			0.508		0.423	0.069		
6	0.785							
7						0.785		
8	0.251				0.749			
9	0.301			0.035	0.261	0.188		
10	0.785							

about 2.4M NPV. At this level of risk, the expected net present value on the single-stage efficient frontier is about 8.9M NPV, versus the expected net present value on the multi-stage curve of about 13.6M NPV, which represents an improvement of 52.8%. In the 'Flat' case, we compare the point with the smallest risk of 3.0M NPV on the single-stage efficient frontier with that with the largest risk of 2.5M NPV on the multi-stage efficient frontier. The expected net present value at the two points is 13.6M NPV (multi-stage) versus 9.6M NPV (single-stage), which represents an improvement of 41.7%. In the 'Steep' case, we compare the point with the smallest risk of 2.6M NPV on the single-stage efficient frontier with that with the largest risk of 1.9M NPV on the multi-stage efficient frontier. The expected net present value at the two points is 20.4M NPV (multi-stage) versus 18.6M NPV (single-stage), which represents an improvement of 9.7%.

As for the single-stage model, we look at the funding strategies at the point of 95% of the maximum expected return (on the efficient frontier the second point down from the maximum expected return point). The funding strategies are presented in table 4 for the 'Normal' data set, and in table A1 and table A2 of appendix A for the 'Steep' and 'Flat' data sets, respectively.

In the 'Normal' case, the initial funding mix consists of 78.5% six-month non-callable debt, and 21.5% five-yea debt callable after one year. After one year the 78.5% six-month debt (that according to the decision rules is refinanced through short-term debt and is for disposition in the second stage) and, if called in certain scenarios, also the five-year callable debt are refunded through various mixes of short-term debt: three-year, five-year and ten-year callable and non-callable debt. In one scenario, labeled '1', in which interest rates fall to a very low level, the multi-stage model resorts to funding with 30-year non-callable debt in order to secure the very low rate for the future. The risk associated with this strategy is 1.85M NPV and the expected net present value is 12.9M NPV. The corresponding (95% of maximum net present value) strategy of the single-stage model, discussed above, has a risk of 3.1M NPV and a net present value of 10.0M NPV. Thus, the multi-stage strategy exhibits 57.4% of the risk of the single-stage strategy and a 29% larger expected net present value.

In the 'Flat' case, the initial funding mix consists of 50.9% three-month non-callable debt and 49.1% six-month non-callable debt. After one year the entire portfolio is refunded through various mixes of short-term, three-year, and ten-year callable and non-callable debt. The multi-stage strategy exhibits 68% of the risk of the single-stage strategy and a 20.6% larger expected net present value. In the 'Steep' case, the initial funding consists of 100% three-month non-callable debt. After one year the portfolio is refunded through various mixes of short-term, three-year and five-year non-callable, and ten-year callable and non-callable debt. The multi-stage strategy exhibits 62% of the risk of the single-stage strategy, and a 4.3% larger expected net present value.

Summarizing, the results demonstrate that multi-stage stochastic programming potentially yields significantly larger net present values at the same or even lower levels of risk, with significantly different funding strategies, compared with single-stage optimization. Using multi-stage stochastic programming for determining the funding of mortgage pools promises to lead, in the average, to significant profits compared with using single-stage funding strategies.

6.4. *Results for duration and convexity*

Funding a mortgage pool by a portfolio of bonds that matches the (negative) value of duration and convexity, the expected net present value of the total of mortgage pool and bonds is invariant to small changes in interest rates. However, duration and convexity give only a local approximation, and the portfolio needs to be updated as time goes on and interest rates change. The duration and convexity hedge is one-dimensional, since it considers only changes of the whole yield curve by the same amount and does not consider different shifts for different maturities. The multi-stage stochastic programming model takes into account multi-dimensional changes of interest rates and considers (via sampling) the entire distribution of possible yield curve developments. In this section we quantify the difference between duration and convexity hedging versus hedging using the single- and multi-stage stochastic programming models. Table 5 gives the initial (first stage) values for duration and convexity (as obtained from the simulation runs) for the mortgage pool and for all funding instruments for each of the three yield curve cases 'Normal', 'Flat', and 'Steep'. In each of the three yield curve cases the mortgage pool exhibits a positive value for duration and a negative value for convexity. All funding instruments have positive values for duration, non-callable bonds exhibit a positive value for convexity, and callable bonds show a negative value for convexity. Note as an exception the positive value for convexity of bond 'y07nc1'.

To both the single-stage and multi-stage stochastic programming models we added constraints that, at any decision point, the duration and convexity of the mortgage pool and the funding portfolio are as close as possible. Maximizing expected return by setting the risk aversion λ to zero, we successively reduced the gap in duration and convexity between the mortgage pool and the funding portfolio. We started from the unconstrained case (the maximum expected return–maximum risk case from the efficient frontiers discussed above) and reduced first the duration gap and subsequently the convexity gap, where we understand as duration gap the absolute value of the difference in duration between the funding portfolio and the mortgage pool, and as convexity gap the absolute value of the difference in convexity between the funding portfolio and the mortgage pool. We will discuss the results with respect to the downside risk measure (expected value of returns below a certain target), as discussed in section 2.9, and also with respect to the standard deviation of the returns. We do not discuss the results for duration and convexity obtained from the single-stage model and focus on the more interesting multi-stage case.

6.4.1. Duration and convexity, multi-stage model. Figures 3 and 4 give the risk–return profile for the 'Normal' case with respect to downside risk and standard deviation, respectively. We compare the efficient frontier (already depicted in figure 2) obtained from minimizing downside risk for different levels of expected return (labeled 'Downside') with the risk–return profile obtained from restricting the duration and convexity gap (labeled 'Delta–Gamma'). For different levels of duration and convexity gap, we maximized expected return. The unconstrained case with respect to duration and convexity is identical to the point on the efficient frontier with the maximum expected return. Figure 3 shows that the downside risk eventually increases when decreasing the duration and convexity gap and was significantly larger than the minimized downside risk on the efficient frontier. We are especially interested in comparing the point with the smallest duration and convexity gap with the point of minimum downside risk on the efficient frontier. The point with the smallest downside risk on the efficient frontier exhibits expected return of 9.5M NPV and a downside risk of 1.5M NPV. The point with the smallest duration and convexity gap has expected return of 7.9M NPV and a downside risk of 3.1M NPV. The latter point is characterized by a maximum duration gap in the first and second stage of 0.5 and of 1.0 in the third stage, and by a maximum convexity gap of 2.0 in the first and second stage and of 4.0

Table 5. Initial duration and convexity.

	Normal		Flat		Steep	
Label	Dur.	Conv.	Dur.	Conv.	Dur.	Conv.
mortg.	3.442	−1.880	2.665	−1.548	2.121	−2.601
m03n	0.248	0.001	0.247	0.001	0.249	0.001
m06n	0.492	0.003	0.491	0.003	0.495	0.003
y01n	0.971	0.010	0.964	0.010	0.982	0.011
y02n	1.882	0.038	1.861	0.038	1.917	0.039
y03n	2.737	0.081	2.686	0.079	2.801	0.084
y03nc1	1.596	−0.384	1.376	−0.186	1.338	−0.235
y05n	4.278	0.205	4.160	0.197	4.378	0.212
y05nc1	1.914	−0.810	1.613	−0.700	1.478	−0.632
y05nc3	3.274	−0.012	3.068	−0.111	3.114	−0.241
y07n	5.622	0.367	5.448	0.351	5.769	0.380
y07nc1	2.378	0.128	1.671	0.959	0.898	0.032
y07nc3	3.406	0.269	3.117	−0.298	3.122	−0.230
y10n	7.335	0.656	7.083	0.624	7.538	0.681
y10nc1	1.555	0.246	1.397	−0.052	0.593	−0.119
y10nc3	3.409	−0.160	3.265	−0.346	3.086	−0.446
y30n	13.711	2.880	13.306	2.743	14.204	3.011

in the third stage. A further decrease of the convexity gap was not possible as it led to an infeasible problem. The actual duration gap in the first stage was −0.5, where the negative value indicates that the duration of the funding portfolio was smaller than that of the mortgage pool, and the actual convexity gap in the first stage was 1.57, where the positive value indicates that the convexity of the funding portfolio was larger than that of the mortgage pool. Comparing the points with regard to their performance, restricting the duration and convexity gap led to a decrease in expected return of 15% and an increase of downside risk by a factor of 2. It is interesting to note that, for the point on the efficient frontier with the smallest risk, the first-stage duration gap was −0.97 and the first-stage convexity gap was 1.75. Looking at the risk in terms of standard deviation of returns, both minimizing downside risk and controlling the duration and convexity gap led to smaller values of standard deviation. In the unconstrained case, the smallest standard deviation was 3.3M NPV obtained at the minimum downside risk point, and the smallest standard deviation in the constrained case (3.8M NPV) was obtained when the duration and convexity gap was smallest.

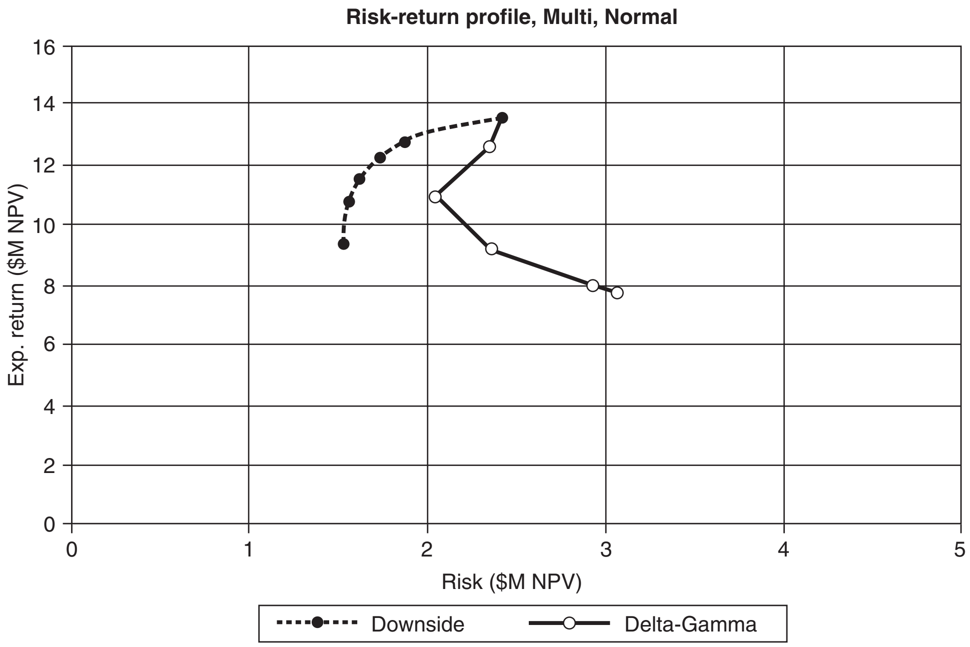

Figure 3. Risk–return profile, multi-stage model, Normal data set, risk as downside risk.

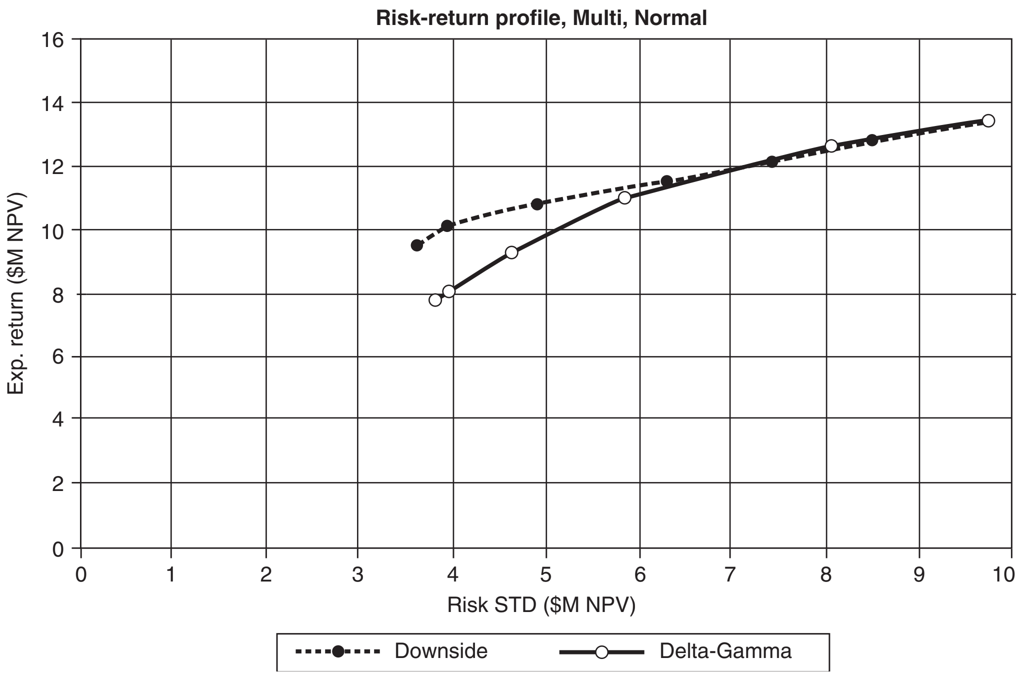

Figure 4. Risk–return profile, multi-stage model, Normal data set, risk as standard deviation.

Table 6 gives the funding strategy for the minimum downside risk portfolio and table 7 the funding strategy for the duration and convexity constrained case at minimum gap. Both strategies use six-month non-callable debt, five-year non-callable debt and five-year debt callable after one year for the initial funding. The minimum downside risk portfolio used in addition five-year debt callable after three years, while the duration and convexity constrained case used seven-year non-callable debt and ten-year non-callable debt. In the second stage, funding differed significantly across the different scenarios for both funding strategies. In the duration and convexity

Table 6. Funding strategy, case Minimum Downside Risk, multi-stage model, Normal data set.

Stage 1	Allocation			
	j1m06n	j1y05n	j1y05nc1	j1y05nc3
	0.266	0.235	0.218	0.281

Stage 2	Allocation							
Scenario	j2m03n	j2y01n	j2y03nc1	j2y05n	j2y10n	j2y10nc1	j2y10nc3	j2y30n
1		0.017		0.148	0.240			0.079
2						0.102	0.165	
3			0.117		0.149			
4	0.266							
5	0.031		0.032		0.237	0.185		
6	0.266							
7					0.014	0.252		
8	0.118				0.300	0.019		0.047
9	0.012				0.201	0.053		
10	0.266							

Table 7. Funding strategy, case Duration and Convexity Constrained, multi-stage model, Normal data set.

Stage 1	Allocation				
	j1m06n	j1y05n	j1y05nc1	j1y07n	j1y10n
	0.122	0.083	0.534	0.239	0.022

Stage 2	Allocation						
Scenario	j2m03n	j2m06n	j2y01n	j2y03nc1	j2y05n	j2y05nc1	j2y07nc1
1	0.466						0.117
2							
3				0.064			
4							
5				0.506	0.043	0.044	
6			0.064				
7							
8	0.567						
9	0.072						
10		0.316	0.340				

Stage 2	Allocation			Call
Scenario	j2y10n	j2y10nc1	j2y10nc3	j1y05nc1
1			0.073	
2	0.011	0.112		
3	0.058			
4	0.152			0.029
5	0.058	0.004		
6	0.059			
7	0.122			
8	0.089			
9			0.051	
10				0.534

constrained case, the multi-stage optimization model resorted to calling five-year debt callable after one year in scenario '4' at a fraction and in scenario '10' at the whole amount. In the minimum risk case, calling of debt other than by applying the decision rules did not happen. In the second stage, interest rates were low in scenarios '1' and '8' and were high in scenarios '4', '6', and '10'. The minimum downside risk strategy tended towards more long-term debt when interest rates were low and towards more short-term debt when interest rates were high. The amounts for each of the scenarios depended on the dynamics of the process and the interest rate distributions. The duration and convexity constrained strategy was, of course, not in the position to take advantage of the level of interest rates, and funding was balanced to match the duration and convexity of the mortgage pool.

The results for the case of the 'Flat' and 'Steep' yield curves are very similar. For the 'Flat' case, the maximum duration gap in the first and second stage was set to 0.5 and to 1.0 in the third stage, and the maximum convexity gap was set to 2.0 in the first and second stage and to 4.0 in the third stage. The duration and convexity gap could not be decreased further since the problem became infeasible. The actual duration gap in the first stage was −0.5, and the actual convexity gap in the first stage was 1.5. In appendix A, table A3 gives the initial funding and the second-stage updates for the minimum downside risk portfolio, and table A4 gives the funding strategy for the duration and convexity constrained case. For the 'Steep' case, the maximum duration gap in the first and second stage was set to 0.5 and was set to 1.0 in the third stage, and the maximum convexity gap was set to 3.0 in the first and second stage and to 6.0 in the third stage. Further decrease made the problem become infeasible. The actual duration gap in the first stage was −0.5, and the actual convexity gap in the first stage was 1.6. In appendix A, table A5 gives the initial funding and the second-stage updates for the minimum downside risk portfolio, and table A6 gives the funding strategy for the duration and convexity constrained case. Again, in both the 'Flat' and 'Steep' case, one could see in the minimum downside risk case a tendency to use longer-term debt when interest rates were low and shorter-term debt when interest rates were high, and in the duration and convexity constrained case, funding was balanced to match the duration and convexity of the mortgage pool.

Summarizing, in each case reducing the duration and convexity gap helped control the standard deviation of net present value but did nothing to reduce downside risk. Multi-stage stochastic programming led to larger than or equal expected net present value at each level of risk (both downside risk and standard deviation of net present value). There may be reasons for controlling the duration and convexity gap in addition to controlling downside risk via the multi-stage stochastic programming model. For example, a conduit might like the results of the stochastic programming model, but not wish to take on too much exposure regarding duration and convexity gap. In the following we will therefore compare runs of the multi-stage model with and without duration and convexity constraints.

Figure 5 sheds more light on the cause of the performance gain of the multi-stage model versus the duration and convexity hedged strategy. The figure presents the duration gap (the difference of the duration of the funding portfolio and the mortgage pool) versus the interest rate (calculated as the average of the yield curve) for each of the second-stage scenarios of the data set 'Normal'. When interest rates are very low, the 95% maximum expected return strategy takes on a significant positive duration gap to lock in the low rates for a long time. It takes on a negative duration gap when interest rates are high, in order to remain flexible, should interest rates fall in the future. The minimum downside risk strategy exhibits a similar pattern, but less extreme. Thus, the multi-stage model makes a bet on the direction interest rates are likely to move, based on the information about the interest rate process. In contrast, the duration and convexity constrained strategy cannot take on any duration gap (represented by the absolute value of 0.5 at which the gap was constrained) and therefore must forsake any gain from betting on the likely direction of interest rates.

6.5. Out-of-sample simulations

In order to evaluate the performance of the different strategies in an unbiased way, true out-of-sample evaluation runs need to be performed. Any solution at any node in the tree obtained by optimization must be evaluated using an independently sampled set of observations.

For the single-stage model this evaluation is rather straightforward. Having obtained an optimal solution from the single-stage model (using sample data set one), we run simulations again with a different seed. Using the new independent sample (data set two), we start the optimizer again, however now with the optimal solution fixed at the values as obtained from the optimization based on sample data set one. Using the second set of observations of data set two we calculate risk and expected returns.

To independently evaluate the results obtained from a K-stage model (having $n = K - 1$ sub-periods), we need K independent sets of K-stage trees of observations. We describe the procedure for $K = 4$. Using data set one we solve the multi-stage optimization problem and obtain an optimal first-stage solution. We simulate again (using a different seed) over all stages to obtain data set two. Fixing the optimal first-stage solution at the value obtained from the optimization based on data set one, we optimize based on data set two and obtain a set of optimal second-stage solutions. We simulate again (with a different seed) to obtain independent realizations for stages three and four, thereby keeping the observations for stage two the same as in data set two, and obtain data set three. Fixing the first-stage decision at the level obtained from the optimization using data set one and all second-stage decisions at the level obtained from the

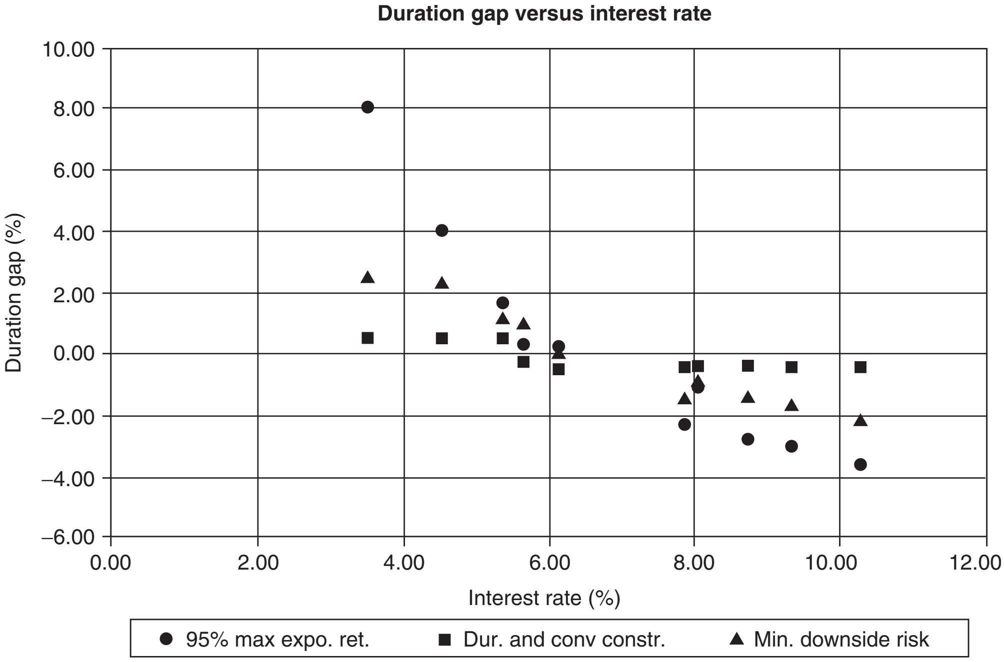

Figure 5. Duration gap versus interest rate, multi-stage, Normal data set.

optimization based on data set two, we optimize again to obtain a set of optimal third-stage decisions. We simulate again (using a different seed) to obtain independent outcomes for stage four, thereby keeping the observations for stage two and three the same as in data sets two and three, respectively, and obtain data set four. Fixing the first-stage decision, all second-stage decisions, and all third-stage decisions at the levels obtained from the optimization based on data sets one, two and three, respectively, we finally calculate risk and returns based on data set four.

The out-of-sample evaluations resemble how the model could be used in practice. Solving the multi-stage model (based on data set one), an optimal first-stage solution (initial portfolio) would be obtained and implemented. Then one would follow the strategy (applying the decision rules) for 12 months until decision stage two arrives. At this point, one would re-optimize, given that the initial portfolio had been implemented and that particular interest rates and prepayments had occurred (according to data set two). The optimal solution for the second stage would be implemented, and one would follow the strategy for four years until decision stage three arrives. At this point, one would re-optimize, given that the initial portfolio and a second stage update had been implemented and that particular interest rates and prepayments had occurred (according to data set three). The optimal solution for the third stage would be implemented, and one would follow the strategy (applying the decision rules) until the end of the horizon (according to data set four). Now one possible path of using the model has been evaluated. Decisions had no information about particular outcomes of future interest rates and prepayments, and were computed based on model runs using data independent from the observed realization of the evaluation simulation. Alternatively, one could simulate a strategy involving re-optimization every month, but this would take significantly more computational effort with likely only little to be gained.

Using the out-of-sample evaluation procedure, we obtain $N = |S|$ out-of-sample simulations of using the model as discussed in the above paragraph, and we are now in the position to derive statistics about out-of-sample expected returns and risk.

Figure 6 presents the out-of-sample efficient frontiers for the 'Normal' data set and downside risk. The figure gives the out-of-sample efficient frontier for the multi-stage model without duration and convexity constraints, the out-of-sample efficient frontier when the duration gap was constrained to be less than or equal to 1.5%, and the out-of-sample efficient frontier for the single-stage model. The out-of-sample evaluations demonstrate clearly that the multi-stage model gives significantly better results than the single-stage model. For example, the point with the maximum expected returns of the multi-stage model gave expected returns of 10.2M NPV and a downside risk of 3.5M NPV. The minimum risk point on the single-stage out-of-sample efficient frontier gave expected returns of 7.6M NPV, and a downside risk of also 3.5M NPV. Thus, at the same level of downside risk the multi-stage model gave 34% higher expected returns. The minimum downside risk point of the multi-stage model gave expected returns of 9.0M NPV and a downside risk of 2.1M NPV. Comparing the minimum downside risk point of the multi-stage model with that of the single-stage model, the multi-stage model had 19.2% larger expected returns at 61% of the downside risk of the single-stage model. The efficient frontier of the duration-constrained strategy was slightly below that without duration and convexity constraints.

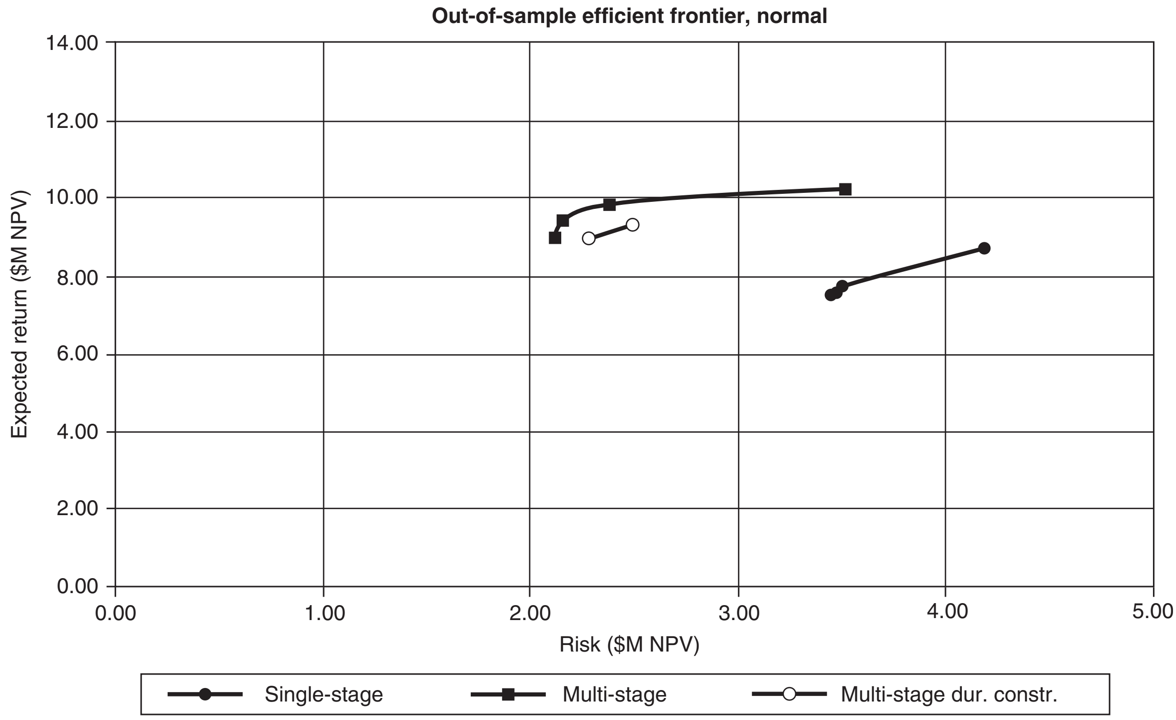

Figure 6. Out-of-sample risk–return profile, multi-stage model, Normal data set, risk as downside risk.

Figure 7 gives the out-of-sample risk–return profiles when measuring risk in terms of standard deviation. In this setting the multi-stage strategy performed significantly better than the single-stage strategy at every level of risk, where the difference was between 17.6% and 23.6%. The duration-constrained strategy did not span as wide a range in risk as the unconstrained strategy. But for the risk points attained by the constrained strategy, the unconstrained strategy achieved a somewhat higher expected return.

The out-of-sample evaluations for the 'Flat' and 'Steep' data sets gave very similar results qualitatively. The results are included in appendix A. For the 'Flat' data set, figure A1 presents out-of-sample efficient frontiers for downside risk and figure A2 the risk–return profile for risk in standard deviations. For the 'Steep' data set, figure A3 presents out-of-sample efficient frontiers for downside risk and figure A4 the risk–return profile for risk in standard deviations for the data set 'Steep'. For both data sets, the multi-stage model gave significantly better results; the downside risk was smaller and expected returns were larger.

6.6. *Larger sample size*

All results discussed so far were obtained from solving a model with a relatively small number of scenarios at each stage, i.e. $|S_2| = 10$, $|S_3| = 20$ and $|S_4| = 20$, with a total of 4000 scenarios at the end of the forth stage. This served well for analysing and understanding the behavior of the multi-stage model in comparison with the single-stage model and with Gamma and Delta hedging. Choosing a larger sample size will improve the obtained strategies (initial portfolio and future revisions) and therefore result in better (out-of-sample simulation) results. Of course, the accuracy of prediction of the models will be improved also. In order to show the effect of using a larger sample size, we solved and evaluated the models using a sample size of 24 000 scenarios, i.e. $|S_2| = 40$, $|S_3| = 30$, and $|S_4| = 20$. The results for the data set 'Normal' are presented in figure 8 for downside risk and in figure 9 for risk as standard deviation. Indeed, one can see improved performance in both smaller risk and larger expected NPV compared with the smaller sample size (compare with figures 6 and 7).

The point with the maximum expected returns for the multi-stage model gave expected returns of 12.9M NPV and a downside risk of 2.4M NPV. The minimum risk point for the single-stage model had expected returns of 9.0M NPV and a downside risk of 2.7M NPV. So, at slightly smaller downside risk the multi-stage model gave expected returns that were 43% higher. The minimum downside risk point of the multi-stage model had expected returns of 10.9M NPV and a downside risk of 1.5M NPV. Comparing the minimum downside risk points of the multi-stage and single-stage models, the multi-stage model has 18.4% larger expected returns at 57% of the downside risk of the single-stage model. Again, the efficient frontier of the duration-constrained strategy was slightly below that without duration and convexity constraints. Measuring risk in terms of standard deviation, results are similar to those when using the small sample size: the multi-stage strategy performed significantly better than the single-stage strategy at every level of risk, with a difference in expected returns between

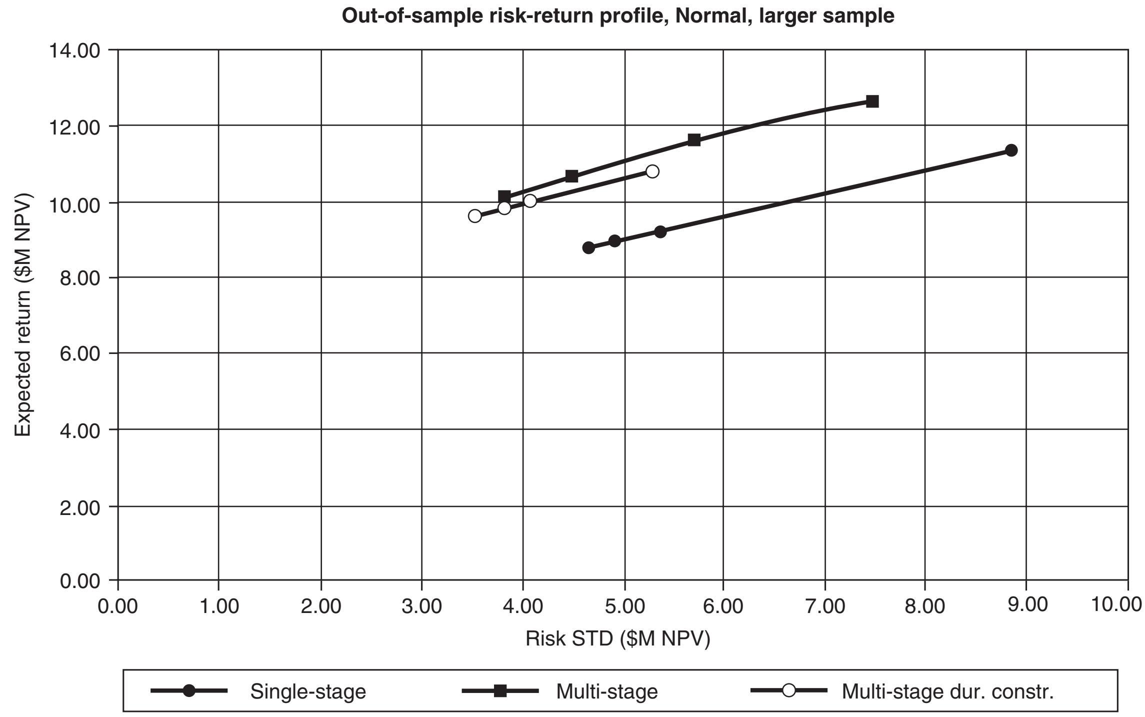

Figure 7. Out-of-sample risk–return profile, multi-stage model, Normal data set, risk as standard deviation.

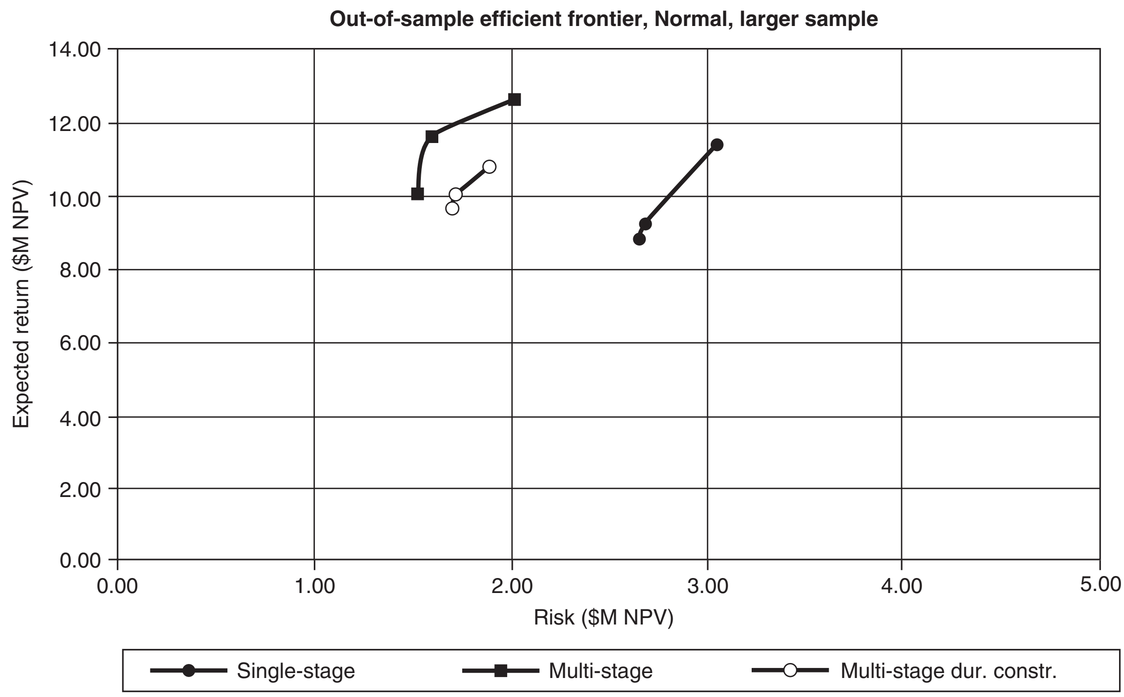

Figure 8. Larger sample, out-of-sample risk–return profile for the multi-stage model, Normal data set, risk as downside risk.

23.3% and 24.0%. Again, the duration-constrained strategy exhibited smaller risk (in standard deviations) at the price of slightly smaller expected returns.

Figure 10 gives a comparison of the multi-stage efficient frontiers predicted (in-sample) versus evaluated out-of-sample. One can see that when using the larger sample size of 24 000, the predicted and the out-of-sample evaluated efficient frontier look almost identical, thus validating the model. It is evident that using larger sample sizes will result in both better performance and a more accurate prediction.

7. Large-scale results

For the actual practical application of the proposed model, we need to consider a large number of scenarios in order to obtain small estimation errors regarding

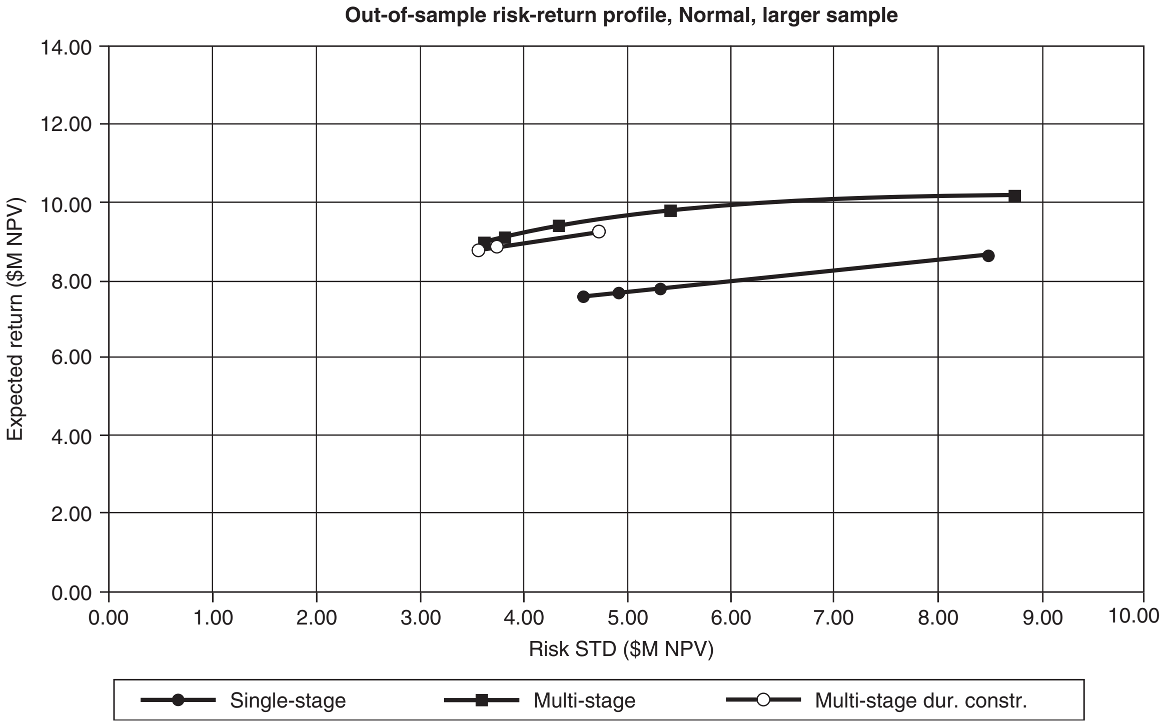

Figure 9. Larger sample, out-of-sample risk–return profile for the multi-stage model, Normal data set, risk as standard deviation.

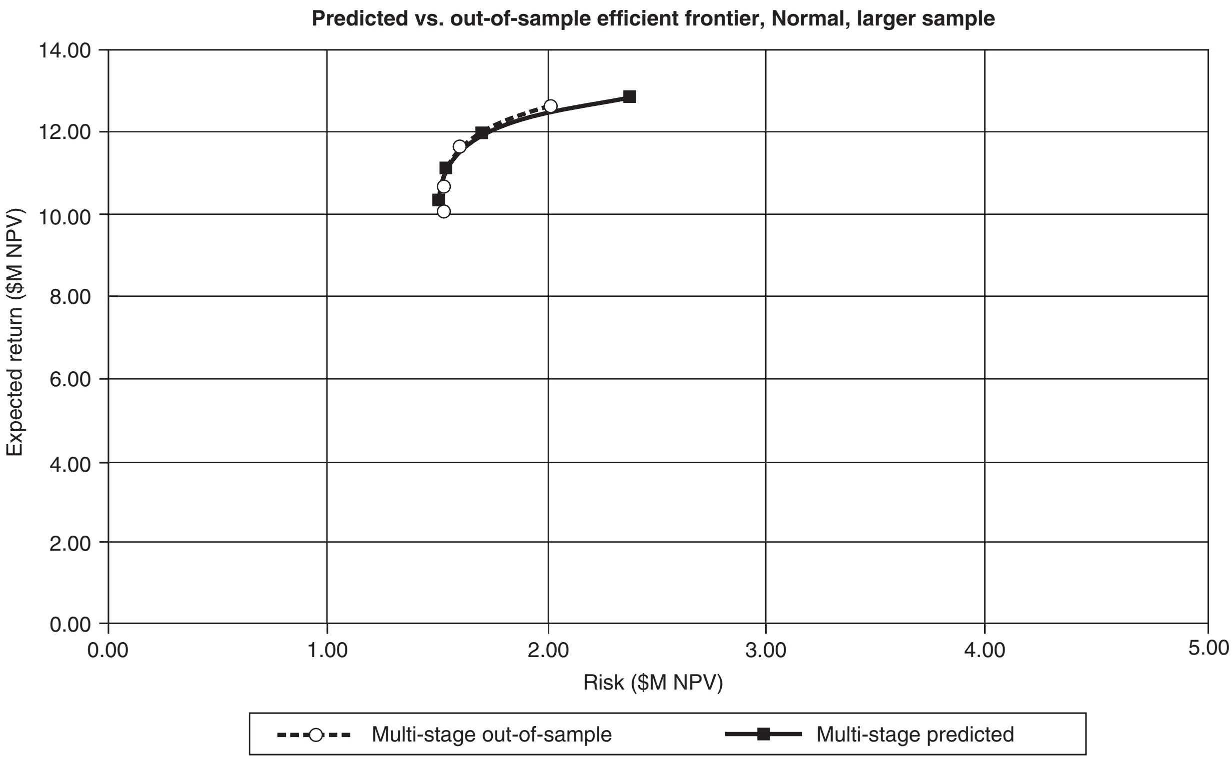

Figure 10. Larger sample, predicted versus out-of-sample efficient frontier for the multi-stage model, Normal data set.

expected returns and risk and accordingly stable results. We now explore the feasibility of solving large-scale models. Table 8 gives measures of size for models with larger numbers of scenarios. For example, model L4 has 80 000 scenarios at the fourth stage, composed of $|S_2| = 40$, $|S_3| = 40$ and $|S_4| = 50$, and model L5 has 100 000 scenarios at the fourth stage composed of $|S_2| = 50$, $|S_3| = 40$ and $|S_4| = 50$; both models have a sufficiently large sample size in each stage.

In the case of problem L5 with 100 000 scenarios the corresponding linear program had 270 154 constraints, 249 277 variables, and 6 291 014 non-zero coefficients.

Table 8. Large-scale models, dimensions.

Model	Scenarios				Problem size		
	Stage 2	Stage 3	Stage 4	Total	Rows	Columns	Non-zeros
L1	10	40	50	20 000	54 034	49 918	1 249 215
L2	20	40	50	40 000	108 064	99 783	2 496 172
L3	30	40	50	60 000	162 094	149 623	3 756 498
L4	40	40	50	80 000	216 124	199 497	5 021 486
L5	50	40	50	100 000	270 154	249 277	6 291 014
L20	200	40	50	400 000	808 201	931 271	14 259 216

Table 9. Large-scale models, solution times.

Model	Scenarios	Simul. time (s)	Direct sol. time (s)	Decomp. sol. time (s)
L1	20 000	688	1389.75	1548.91
L2	40 000	1390	6144.56	
L3	60 000	2060	14 860.54	5337.76
L4	80 000	2740	28 920.69	
L5	100 000	3420	48 460.02	9167.47
L20	400 000			41 652.28

Table 9 gives the elapsed times for simulation and optimization, obtained on a Silicon Graphics Origin 2000 workstation.

While the Origin 2000 at our disposition is a multi-processor machine with 32 processors, we did not use the parallel feature, and all results were obtained using single processor computations. For the direct solution of the linear programs, we used CPLEX (1989–1997) as the linear programming optimizer. We contrast the results to using DECIS (Infanger 1989–1999), a system for solving stochastic programs, developed by the author. Both CPLEX and DECIS were accessed through GAMS (Brooke *et al.* 1988). DECIS exploited the special structure of the problems and used dual decomposition for their solution. The problems were decomposed into two stages, breaking between the first and the second stage. The simulation runs for model L5 took less than an hour. The elapsed solution time solving the problem directly was 13 hours and 28 minutes. Solving the problem using DECIS took significantly less time, 2 hours and 33 minutes. Encouraged by the quick solution time using DECIS, we generated problem L20 with 400 000 scenarios (composed of $|S_2| = 200$, $|S_3| = 40$ and $|S_4| = 50$) and solved it in 11 hours and 34 minutes using DECIS. Problem L20 had 808 201 constraints, 931 271 variables, and 14 259 216 non-zero coefficients.

Using parallel processing, the simulation times and the solution times could be reduced significantly. Based on our experiences with parallel DECIS, using six processors we would expect the solution time for the model L5 with 100 000 scenarios to be less than 40 minutes, and using 16 processors one could solve model L20 with 400 000 scenarios in about one hour. We actually solved a version of the L3 model with 60 000 scenarios, composed of $|S_2| = 50$, $|S_3| = 40$ and $|S_4| = 30$, in less than 5 minutes using parallel DECIS on 16 processors.

8. Summary

The problem of funding mortgage pools represents an important problem in finance faced by conduits in the secondary mortgage market. The problem concerns how to best fund a pool of similar mortgages through issuing a portfolio of bonds of various maturities, callable and non-callable, and how to refinance the debt over time as interest rates change, prepayments are made and bonds mature. This paper presents the application of stochastic programming in combination with Monte Carlo simulation for effectively and efficiently solving the problem.

Monte Carlo simulation was used to estimate multiple realizations of the net present value of all payments when a pool of mortgages is funded initially with a single bond, where pre-defined decision rules were applied for making decisions not subject to optimization. The simulations were carried out in monthly time steps over a 30-year horizon. Based on a scenario tree derived from the simulation results, a single-stage stochastic programming model was formulated as a benchmark. A multi-stage stochastic programming model was formulated by splitting up the planning horizon into multiple sub-periods, representing the funding decisions (the portfolio weights and any calling of previously issued callable bonds) for each sub-period, and applying the pre-defined decision rules between decision points.

In order to compare the results of the multi-stage stochastic programming model with hedging methods typically used in finance, the effective duration and convexity of the mortgage pool and of each funding instrument was estimated at each decision node, and constraints bounding the duration and convexity gap to close to zero were added (at each node) to the multi-stage model to approximate a duration and convexity hedged strategy.

An efficient method for obtaining an out-of-sample evaluation of an optimal strategy obtained from solving a K-stage stochastic programming model was presented, using K independent (sub-)trees for the evaluation.

For different data assumptions, the efficient frontier of expected net present value versus (downside) risk obtained from the multi-stage model was compared with that from the single-stage model. Under all data assumptions, the multi-stage model resulted in significantly better funding strategies, dominating the single-stage model at every level of risk, both in-sample and by evaluating the obtained strategies via out-of-sample simulations. Also, for all data assumptions, the out-of-sample simulations demonstrated that the multi-stage stochastic programming model dominated the duration and convexity hedged strategies at every level of risk. Constraining the duration and convexity gap reduced risk in terms of the standard deviation of net present value at the cost of a smaller net present value, but failed in reducing the downside risk.

The results demonstrate clearly that using multi-stage stochastic programming results in significantly larger profits, both compared with using single-stage optimization models and with using duration and convexity hedged strategies. The multi-stage model is better than the single-stage model because it has the option to revise the funding portfolio according to changes in interest rates and pre-payments, therefore reflecting a more realistic representation of the decision problem. It is better than the duration and convexity hedged strategies because it considers the entire distribution of the yield curve represented by the stochastic process of interest rates, compared with the much simpler hedge against a small shift of the entire yield curve as used in the duration and convexity hedged strategies.

Small models with 4000 scenarios and larger ones with 24 000 scenarios were used for determining the funding strategies and the out-of-sample evaluations. The out-of-sample efficient frontiers of the larger models were shown to be very similar to the (in-sample) predictions, indicating a small optimization bias. The larger models were solved in a very short (elapsed) time of a few minutes. Large-scale models with up to 100 000 scenarios were shown to solve in a reasonable elapsed time using decomposition on a serial computer, and in a few minutes on a parallel computer.

Acknowledgements

Funding for research related to this paper was provided by Freddie Mac. The author is grateful to Jan Luytjes for many valuable discussions on the subject. Editorial comments by three anonymous referees and by John Stone on previous versions of this paper helped enhance the presentation.

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Appendix A: Tables and graphs for data sets "Flat" and "Steep"

Table A1. Funding strategy from the multi-stage model, case 95% of maximum expected return, Flat data set.

Stage 1	Allocation						
	j1m03n	j1m06n					
	0.509	0.491					
Stage 2	**Allocation**						
Scenario	j2m03n	j2y03n	j2y03nc1	j2y10n	j2y10nc1	j2y10nc3	j2y30n
1				0.397			0.603
2					1.000		
3	0.251	0.590		0.011		0.148	
4	1.000						
5			0.514	0.248	0.238		
6	1.000						
7	0.984				0.016		
8	0.620			0.380			
9	0.521			0.151	0.328		
10	1.000						

Table A2. Funding strategy from the multi-stage model, case 95% of maximum expected return, Steep data set.

Stage 1	Allocation						
	j1m03n						
	1.000						
Stage 2	**Allocation**						
Scenario	j2m03n	j2y03n	j2y03nc1	j2y05n	j2y10n	j2y10nc1	j2y30n
1		0.038		0.296	0.544		0.122
2	0.717		0.283				
3			1.000				
4	1.000						
5			0.745		0.085	0.170	
6	1.000						
7	1.000						
8	0.609				0.272	0.119	
9	0.973				0.027		
10	1.000						

Table A3. Funding strategy from the multi-stage model, case Minimum Downside Risk, Flat data set.

Stage 1	Allocation	
	j1m06n	j1y05nc3
	0.698	0.302

Stage 2	Allocation					
Scenario	j2m03n	j2m06n	j2y03n	j2y03nc1	j2y05n	j2y05nc3
		0.046			0.418	0.052
2						
3	0.201		0.292			
4	0.698					
5	0.119			0.145		
6	0.698					
7	0.603					
8	0.333					
9	0.302					
10	0.698					

Stage 2	Allocation			
Scenario	j2y10n	j2y10nc1	j2y10nc3	j2y30n
1	0.129			0.053
2		0.698		
3	0.039	0.039	0.127	
4				
5	0.221	0.214		
6				
7		0.095		
8	0.280	0.085		
9	0.095	0.301		
10				

Table A4. Funding strategy from the multi-stage model, case Duration and Convexity Constrained, Flat data set.

Stage 1	Allocation			
	j1m06n	j1y05n	j1y05nc1	j1y07n
	0.448	0.225	0.201	0.125

Stage 2	Allocation							Call
Scenario	j2m03n	j2m06n	j2y02n	j2y03n	j2y03nc1	j2y10n	j2y10nc1	j1y05nc1
1	0.567					0.082		
2	0.102						0.346	
3	0.370			0.078				
4		0.251	0.033			0.164		
5	0.018				0.631			
6		0.397				0.051		
7						0.036	0.412	
8	0.634					0.016		
9	0.448							
10	0.277	0.372						0.201

Table A5. Funding strategy from the multi-stage model, case Minimum Downside Risk, Steep data set.

Stage 1	Allocation
	j1m03n
	1.000

Stage 2 Scenario	j2m03n	j2y01n	j2y03n	j2y03nc1	j2y05n	j2y10n	j2y10nc1	j2y30n
1		0.299			0.389	0.215		0.097
2	0.644			0.356				
3				1.000				
4	1.000							
5				0.734		0.088	0.178	
6	1.000							
7	1.000							
8	0.600		0.005			0.246	0.149	5.126743E-4
9	0.958					0.042	8.23454E-4	
10	1.000							

Table A6. Funding strategy from the multi-stage model, case Duration and Convexity Constrained, Steep data set.

Stage 1	Allocation		
	j1y01n	j1y03n	j1y10n
	0.682	0.305	0.013

Stage 2 Scenario	j2m03n	j2y01n	j2y02n	j2y03n	j2y03nc1	j2y10n	j2y30n
1	0.598						0.084
2		0.053		0.629			
3	0.049	0.633					
4				0.531		0.151	
5					0.682		
6		0.158		0.524			
7		0.500				0.182	
8	0.682						
9	0.682						
10		0.392	0.289				

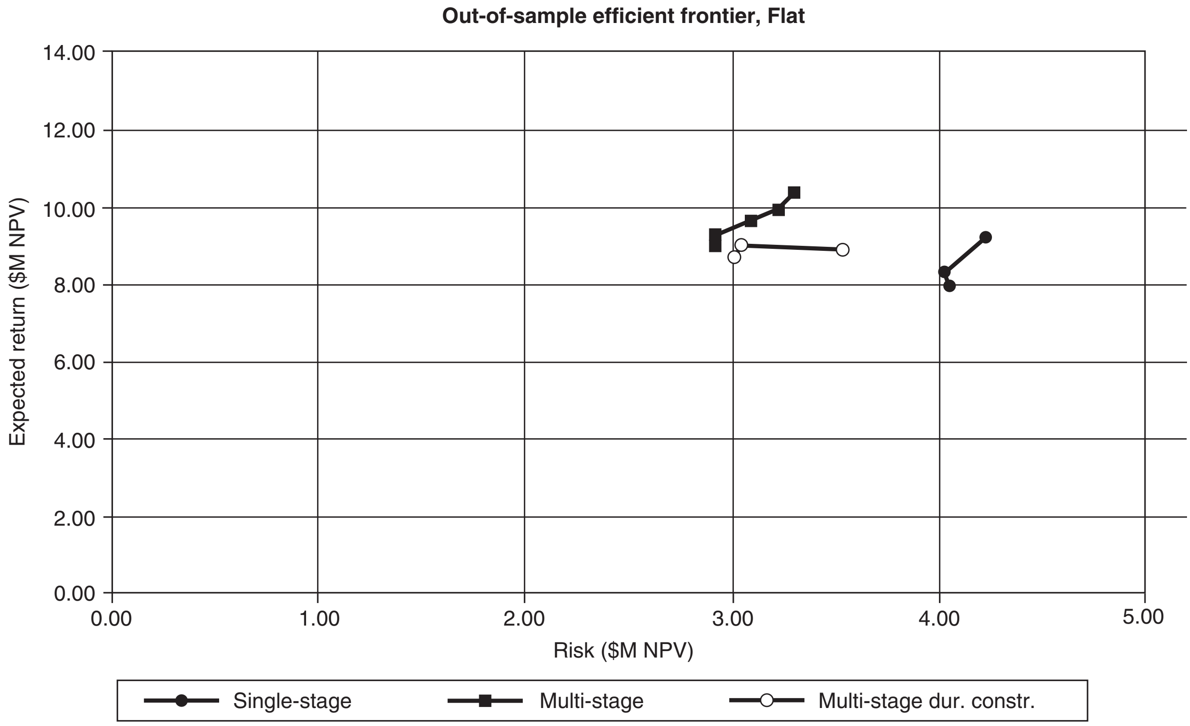

Figure A1. Out-of-sample risk–return profile for the multi-stage model, Flat data set, risk as downside risk.

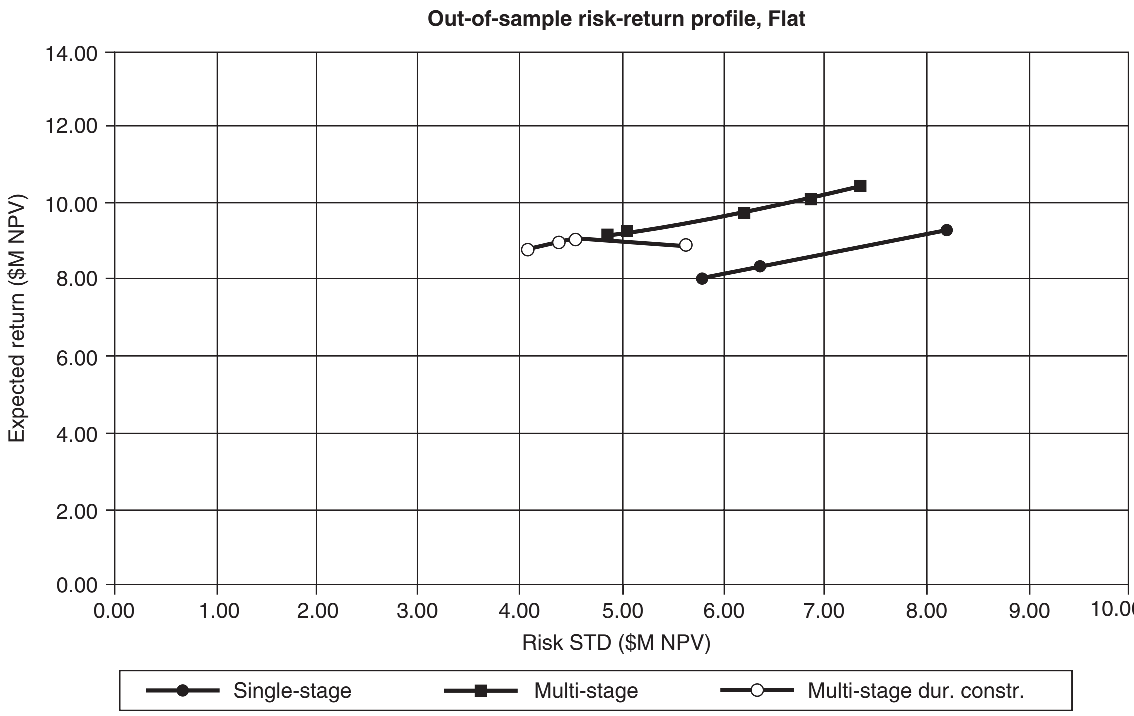

Figure A2. Out-of-sample risk–return profile for the multi-stage model, Flat data set, risk as standard deviation.

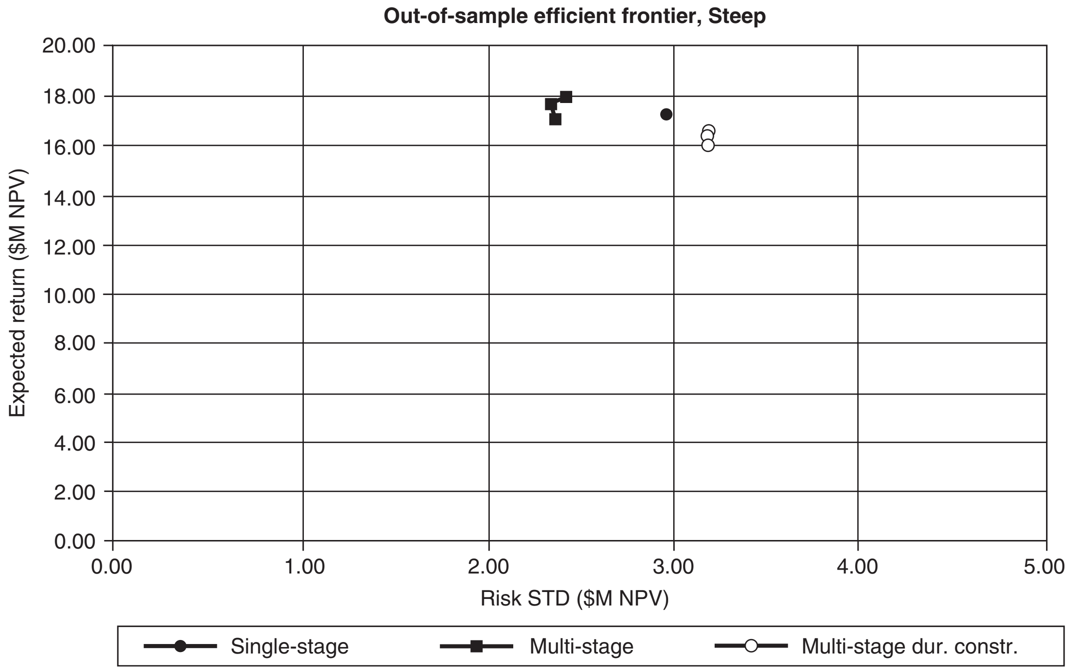

Figure A3. Out-of-sample risk–return profile for the multi-stage model, Steep data set, risk as downside risk.

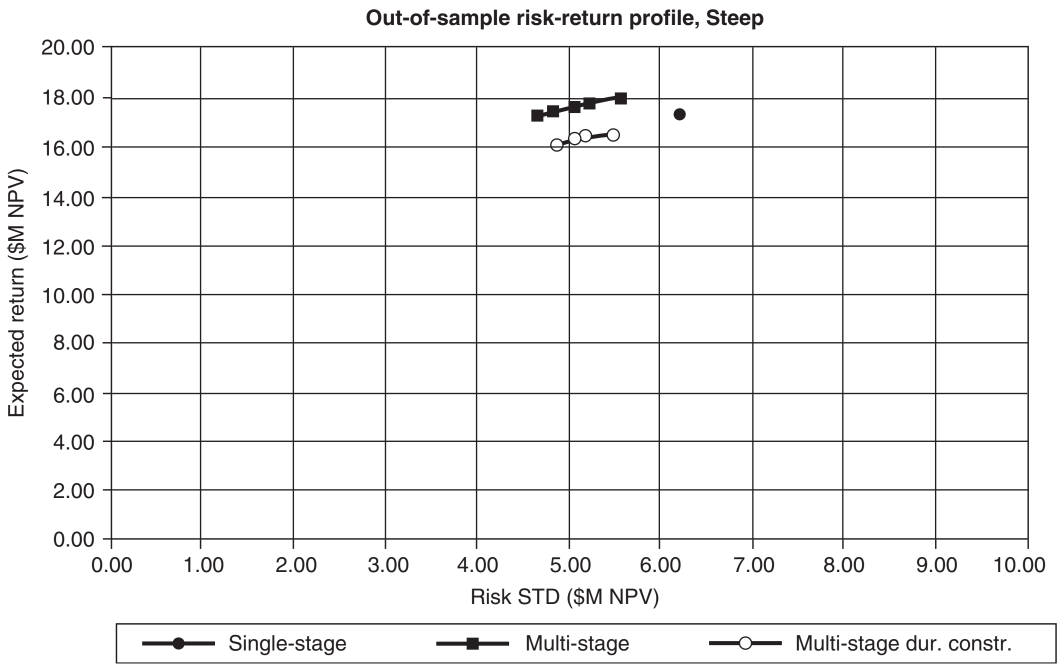

Figure A4. Out-of-sample risk–return profile for the multi-stage model, Steep data set, risk as standard deviation.