MATH TOOLS - DIFF EQ 1

1. Examples of things that use differential equations - WHY ARE WE LEARNING THIS??? ⇒ BECAUSE EVERYTHING IN BIO IS A DYNAMICAL SYSTEM! THEY EVOLVE OVER TIME!

→ Many examples in neuroscience: 1. action potentials are modeled w/ diff eqs (HH model is a particularly famous example)
2. oscillatory activity is modeled w/ diff. eq.
3. RC-circuit model is a diff. model
4. neural networks can be thought of as multi-dimensional ODE systems

→ Examples outside of neuroscience: 1. predator prey models / models of spreading infection - simplest models are ODEs.
2. (Firefly) synchrony → understood w/ ODEs
3. Models of growth/decay (half-life, doubling times) related to eigenvalues!
4. cell cycle models - mitosis
5. hysteresis - actually sol'ns to ODEs.
6. chemical rxns

Common eqs:
HH eqs, Schrodinger eq, Maxwell eqs.

TODAY - we will cover a few of those models: (1) exponential growth/decay, (2) RC-circuit, (3) population model, (4) chemical reactions, (5) lasers, (6) hysteresis examples.

~~GOAL - develop some tools & familiarity w/ hopefully relevant examples~~

2. What exactly is a differential equation?

⇒ An equation that relates some function with its derivative(s).

Simplest example: $\frac{d}{dt}[f(t)] = m$ ⇒ ALSO WRITTEN: $\frac{df}{dt} = m$, $\dot{f} = m$

This one is easy to interpret → derivative of f is a constant, $f(t) = mt + c$
$f(t)$ is a line! ⇒ need to know more about "initial conditions" to know what c is

⇒ Diff eqs come in many flavors:

1. $\boxed{\text{Linear}}$ ($\dot{P} = \alpha P$) growth vs. nonlinear ($\dot{P} = P^2$) heat eq
2. $\boxed{\text{Ordinary}}$ ($\dot{P} = \sin(P)$) vs. Partial ($\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$)
3. $\boxed{\text{Continuous}}$ ($\dot{P} = \alpha P$) vs. "Discrete (iterative maps")
$x_{n+1} = r x_n (1 - x_n)$ ← logistic eq. exhibits chaos!

FOCUS ON THESE

③ ~~How~~ How do you solve these things?

→ We will cover 3 main approaches:

① analytical - good to know (good for solid kind of thing), but rarely used in practice (minus separation of variables, sometimes)

② qualitative - important for developing intuition of a system.

③ numerical - most often used in practice.

⇒ the approach you choose can depend on many things, like

- what you care about (exact solutions? long term behavior/fixed points?)

We will go over examples of all these methods → ① Analytic ② Analytic ③ Qualitative

TODAY - Chalk-talk, lots of examples, more "applied mathy" & less "data analysis"

EXAMPLE 1 - exponential growth model. this may be a warm-up for some people. practice analytical solution-finding.

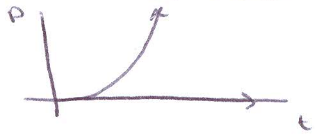
ODE: $\frac{dP}{dt} = \alpha P$

P - dependent var. (Population = # of animals)
t - independent var. (time)
α - scalar value; parameter controlling rate of change

SAY THIS FIRST ↓

WORD MODEL: "rate of growth of population is proportional to the size of the population" (ignoring space/nutrient constraints)

→ use this example as a start to understanding the analytical tool I will go over (separation of variables), since we already know what the solutions look like:



ⓐ Easily identify the trivial solution: $\boxed{P=0}$

CHECK: $\frac{dP}{dt} = \frac{d}{dt}[0] = 0$
$\alpha[0] = 0$ same ✓

ⓑ To find nontrivial solution - two options: ① solve, ② guess

Guess - function whose derivative is the same, multiplied by a constant

$\Rightarrow P(t) = e^{\alpha t}$

CHECK: $\underbrace{\frac{d}{dt}[e^{\alpha t}]}_{\text{Chain rule! (Review?)}} = \alpha e^{\alpha t}$
$\alpha P = \alpha e^{\alpha t}$ same!

⇒ NOTE: could have also been $P(t) = [\text{any constant}]\, e^{\alpha t}$

Thus, the most general solution is:

$$P(t) = Ae^{\alpha t} \quad \text{[general solution]}$$

$\uparrow$ any scalar

Generally, $A = P_0$, because $P(0) = Ae^{\alpha(0)} = A$.

If we know the value of P_0, then $P(t) = P_0 e^{\alpha t}$ is called a particular solution.

(c) What if we don't want to guess?

TOOL ONE: Separation of variables. Sometimes you actually use this!

$$\frac{dP}{dt} = \alpha P$$

$$\int \frac{dP}{P} = \int \alpha \, dt \qquad \int \frac{1}{x} = \ln(x)$$

$$\ln(P) + C = \alpha t + C'$$

$$e^{\ln(P)} = \alpha t + C \qquad e^{\ln(x)} = x$$

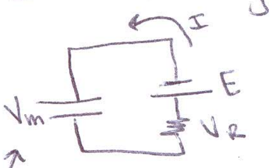
$$P(t) = e^{\alpha t + C} = e^{\alpha t} e^{C} = \overset{P_0}{A} e^{\alpha t}$$

$$P(t) = P_0 e^{\alpha t}$$

NOTE $\Rightarrow$ equations must separable!

$$\frac{dy}{dt} = g(t)\, h(y) \qquad \dot{y} = yt \ \checkmark \qquad \dot{y} = y + t \ \times$$

Example 2 - Analytical Solution to RC-circuit



want to know V_m over time

$$E = V_m + V_R \qquad Q = CV_m \qquad V_R = IR$$

$$\frac{d}{dt}[Q = CV_m] \Rightarrow I = C\frac{dV_m}{dt}$$

$$E = V_m + V_R$$
$$E = V_m + IR$$
$$\frac{E - V_m}{R} = I$$

$$\frac{E - V_m}{R} = C\frac{dV_m}{dt}$$

$$\Rightarrow \frac{dV_m}{dt} = \frac{E - V_m}{RC} \qquad (\dot{P} = \alpha P)$$

$$\int \frac{dV_m}{E - V_m} = \int \frac{dt}{RC}$$

$$-\ln(E - V_m) = \frac{t}{RC} + V_0$$

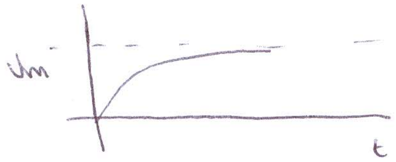
$$E - V_m = V_0 e^{-t/RC} \Rightarrow V_m(t) = E - V_0 e^{-t/RC}$$

$\Rightarrow V_m(t) = E - V_0 e^{-t/RC}$ $\tau = RC$

$= E - V_0 e^{-t/\tau}$ $V_m(0) = 0$

$V_m(0) = E - V_0 e^{-(0)/\tau} = E - V_0 = 0 \Rightarrow E = V_0$

$V_m(t) = E - E e^{-t/\tau} \Rightarrow E(1 - e^{-t/\tau})$



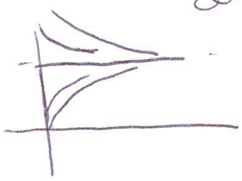
$\Rightarrow$ can get intuition for this based on differential. $\rightarrow$ SLOPE FIELDS

$\frac{dV_m}{dt} = \frac{E - V_m}{RC}$

$V_m = 0$ ($V_m < E$) $\rightarrow \dot{V}_m = \frac{E}{RC} > 0$, so $V_m \uparrow$

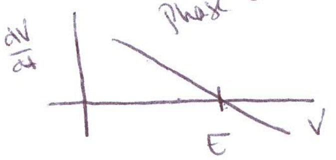
$V_m = E \rightarrow \dot{V}_m = \frac{E - E}{RC}$, so $V_m \leftrightarrow$

$V_m > E \rightarrow \dot{V}_m < 0$, so $V_m \downarrow$

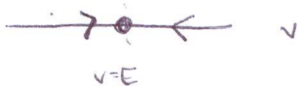


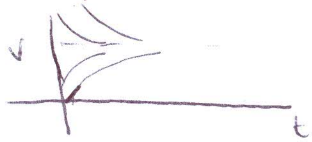
In $\frac{E - V_m}{RC} \rightarrow$ no t

phase line "qualitative analysis"



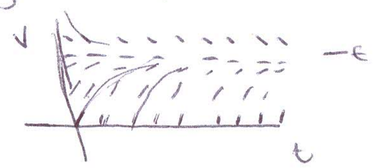
$\Rightarrow$ "phase line"





$\frac{dV}{dt}(V,t)$

slope fields:



quiver

$\frac{dV}{dt}(V,t) = \frac{E - V_m}{RC}$

LOGISTIC MODEL

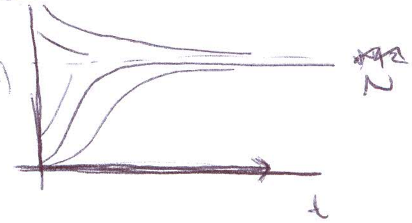
$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$ carrying capacity

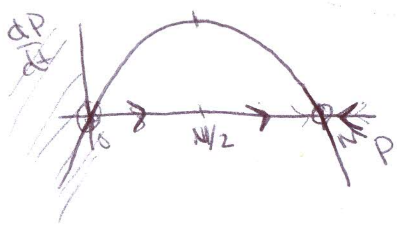
$\frac{dP}{dt} \approx kP$ when P is small

$P < N$, $\frac{dP}{dt} > 0$

$P > N$, $\frac{dP}{dt} < 0$

$P = N$, $\frac{dP}{dt} = 0$





$kP - \frac{kP^2}{N}$

$f(P) = kP - \frac{kP^2}{N}$

$f'(P) = k - \frac{2kP}{N} = 0$

$\frac{2kP}{N} = k$

$2P = N$

$\boxed{P = \frac{N}{2}}$

$\rightarrow$ ① compute f.p. $[P$ s.t. $\frac{dP}{dt} = 0]$ P^*

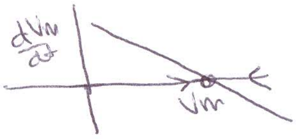
$\rightarrow$ ② investigate phase portrait $\frac{dP}{dt} = f(P)$

$f'(P^*) < 0 \rightarrow P^*$ is stable

$f'(P^*) > 0 \rightarrow P^*$ is unstable

One way to make this easy:

pex

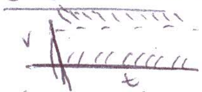


$\Leftarrow$ one stable solution

"QUALITATIVE ANALYSIS"

PHASE LINE

SLOPE FIELDS



(invariant across t)

for various V & t_1 what is $\frac{dV}{dt}$?

(3) ~~[illegible]~~ logistic model as an example of qualitative analysis

$\dot{x} = \sin(x)$ ← many times it is a real pain to analytically solve something.

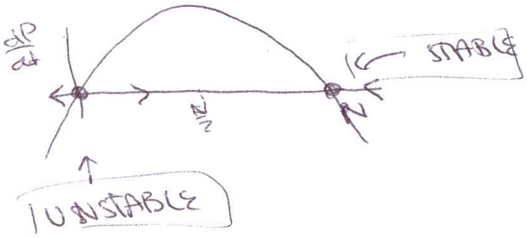
biological model example:

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P = kP \boxed{- \frac{kP^2}{N}} \rightarrow \frown$$

assumptions: $\frac{dP}{dt} \approx kP$ if P is small

if $P > N$, $\frac{dP}{dt} < 0$

when does $\frac{dP}{dt} = 0$? $\rightarrow$ $P = 0$ & $P = N$

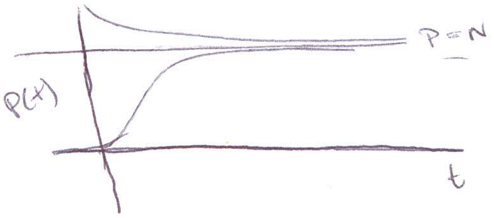


$$f(p) = kP - \frac{kP^2}{N}$$

$$f'(P) = k - \frac{2k}{N}P = k\left(1 - \frac{2P}{N}\right) = 0$$

$$\frac{2P}{N} = 1$$

$$P = \frac{N}{2}$$



~~[illegible]~~

To be more rigorous/quantitative about stability: [CLASSIFICATION of EQUILIBRIA]

- stable/sink: if a sol'n starts near it, it will go toward it
- unstable/source: if sol'ns that start near y_0 go towards y_0 as ~~t~~.

$\Rightarrow$ determine equilibria from $f(y)$ $\Rightarrow$ LINEARIZATION THEOREM

(1) if $f'(y_0) < 0$, y_0 is a sink

(2) if $f'(y_0) > 0$, y_0 is a source

(3) if $f'(y_0) = 0$, need more info (partially stable?)

$f(y) \Rightarrow$ tells us behavior of (?) linear approximation to f near y_0

~~(4) Another example of qualitative analysis (biology)~~

linear stability analysis $\longrightarrow$

L.S.A.

Let x^* be a fixed point, $\underbrace{\eta(t) = x(t) - x^*}$ be a small perturbation

↳ does this grow or decay?

⇒ write ODE for $\eta(t)$ → $\dot{\eta} = \frac{d\eta}{dt} = \frac{d}{dt}(x(t) - x^*) = \dot{x}$

$$\Rightarrow \dot{\eta} = \dot{x} = f(x) = f(x^* + \eta) = \underbrace{f(x^*)}_{=0} + \eta f'(x^*) + O(\eta^2)$$

$$\boxed{\dot{\eta} \approx \eta f'(x^*)}$$ ← linear equation in η, linearization about x^*

⇒ η grows exponentially if $f'(x^*) > 0$, shrinks exponentially if $f'(x^*) < 0$

if $f'(x^*) = 0$ → need to find out more $\boxed{\dot{x} = -x^3}$ (stable)

⇒ $|f'(x^*)|$ → characteristic time scale

(4) ~~[illegible]~~ Relating this idea to "energy wells" → related to potential energy

$f(x) = -\frac{dV}{dx} = \frac{dx}{dt}$ $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$ $\boxed{\text{chain rule}}$ V → potential, defined by $f(x) = -\frac{dV}{dx}$

$f(x) = -c$

$$\frac{dV}{dt} = \left(\frac{dV}{dx}\right)\left(-\frac{dV}{dx}\right) = -\left(\frac{dV}{dx}\right)^2 \leq 0$$ → decrease along trajectories

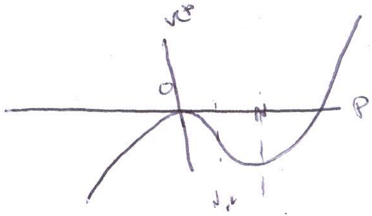
⇒ minima correspond to stable f.p.
maxima correspond to unstable f.p.

BACK to OLD MODEL: $\frac{dP}{dt} = k(1 - \frac{P}{N})P = kP - \frac{kP^2}{N} = -\frac{dV}{dP}$

$$\Rightarrow \frac{dV}{dP} = -kP + \frac{kP^2}{N}$$

$$\int dV = \int\left(-kP + \frac{kP^2}{N}\right)$$

$$V = \frac{-kP^2}{2} + \frac{kP^3}{3N} =$$



(4) Another example: Chemical Rxn (autocatalysis)

Model this system: $A + \boxed{X} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 2\boxed{X}$

① Law of mass action: rate of rxn is proportional to concentration
② A is in SURPLUS

$$\Rightarrow \frac{d[X]}{dt} = +k_1[A][X] - k_2[X]^2$$

$$\Rightarrow \frac{dx}{dt} = k_1 a x - k_2 x^2$$

① find f.p. $0 = k_1 a x - k_2 x^2$

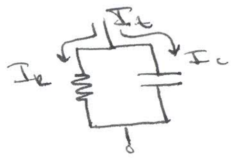
$k_1 a x = k_2 x^2$

$k_1 a = k_2 x$

$x^* = 0$ ← stable
$x^* = \frac{k_1}{k_2} a$ ← unstable

$f(x) = k_1 a x - k_2 x^2$
$f'(x) = k_1 a - 2k_2 x$
$f'(0) = k_1 a > 0$
$f'(\frac{k_1 a}{k_2}) = k_1 a - 2k_2(\frac{k_1}{k_2}a)$
$= k_1 a - 2k_1 a = -k_1 a < 0$

EXAMPLE PROBLEM – LEAKY INTEGRATE & FIRE MODEL



$I_t = I_R + I_c$

$I_R = \frac{V}{R}$

$I_c \Rightarrow Q = CV$

$I = C\frac{dV}{dt}$

$I_t = \frac{V}{R} + C\frac{dV}{dt}$

$C\frac{dV}{dt} = I_{app} - \frac{V}{R}$

$\overset{\tau_m}{RC}\frac{dV}{dt} = I_{app}R - V$

$\tau\frac{dV}{dt} = I_{app}R - V$

TO SOLVE:

$\int \frac{dV}{IR - V} = \int \frac{dt}{\tau}$

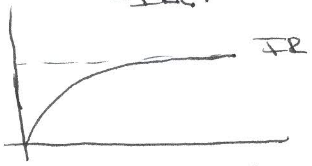
$-\ln(IR - V) = \frac{t}{\tau} + C$

$IR - V = Ae^{-t/\tau}$

$V(t) = IR - Ae^{-t/\tau}$ → $V(0) = IR - A = 0$

$A = IR$

$= IR(1 - e^{-t/\tau})$



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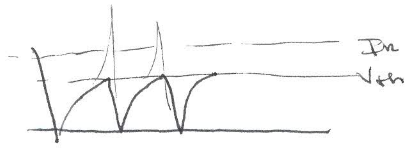
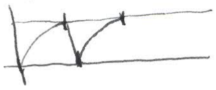


$V_{th} < IR$

If $V_{thresh} \not< IR$, then $V \to 0$.

What is the firing rate of this model neuron?
(How does this vary as a function of I & V_{th}?)

FIND ISI → $FR = \frac{1}{ISI}$



⇒ $V(t) = IR(1 - e^{-t/\tau})$

~~Vth = IR(1-e~~

$V(t^*) = \left[IR(1 - e^{-t^*/\tau}) = V_{th}\right]$ SOLVE.

$1 - e^{-t^*/\tau} = \frac{V_{th}}{IR}$

$1 - \frac{V_{th}}{IR} = e^{-t^*/\tau}$

$\ln(1 - \frac{V_{th}}{IR}) = \frac{-t^*}{\tau}$

$-\tau\ln(1 - \frac{V_{th}}{IR}) = t^*$

$FR = \frac{1}{t^*}$

① ⇒ as $V_{th} \to IR$, $[1 - \frac{V_{th}}{IR}] \to 0$, so ISI ↑ (FR↓)

② ⇒ as I ↑, then $\frac{V_{th}}{IR}$ ↓, $1 - \frac{V_{th}}{IR}$ ↑, so ISI ↓ (FR↑)

2-dimensional systems (linear)

These are systems of the form:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

} NOTE: THESE ARE LINEAR!

How do we solve these systems? (Numerical integration usually)

But we can get a handle on these systems analytically too!

⇒ Search for solutions of this form: $\vec{x}(t) = e^{\lambda t}\vec{v}$

$$x(t) = e^{\lambda t} v_1 \qquad y(t) = e^{\lambda t} v_2$$

Rewrite in matrix form:

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{z}$$

CHECK:
$\dot{x} = ax + by$
$\dot{y} = cx + dy$ ✓

If $x(t) = e^{\lambda t} v_1$ then $\dot{x} = \lambda e^{\lambda t} v_1$
$\dot{y} = \lambda e^{\lambda t} v_2$

$$\Rightarrow \begin{bmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \end{bmatrix}$$

$$\lambda e^{\lambda t}\vec{v} = A\vec{v}e^{\lambda t}$$

$$\boxed{\lambda \vec{v} = A\vec{v}} \longrightarrow$$ eigenvector - eigenvalue equation
⇒ FIND eigenvalues of A, which will determine the stability! (instability)!

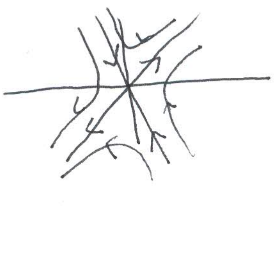
$$A\vec{v} - \lambda\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0 \quad \Rightarrow A - \lambda I \text{ is not invertible}$$

$$\det(A - \lambda I) = 0$$

⇒ eigenvectors determine the direction of straight line solutions!

$$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0$$



$$(a-\lambda)(d-\lambda) - bc = 0$$

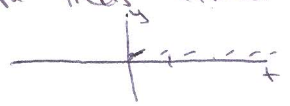
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

NUMERICAL METHODS → real-life, use ode-solvers built in (MUCH better) → DON'T use Euler in practice

(1) Euler's method: ⇒ related to slope fields (finds solution by following slope field)

$\frac{dy}{dt} = f(t, y)$ $\quad y(t_0) = y_0$



(1) issues w/ stability & uniqueness

← to make jump just need to know derivative!

w/ initial point

$y(t_0) = y_0$, find y_1

$y_1 =$ ~~[illegible]~~ $\quad \nearrow \left.\frac{dy}{dt}\right|_{t=t_0} \to f(y(t_0))$ $\quad f(y_0)$

$y_1 = y(t_1) = y(t_0 + \Delta t) \quad = y(t_0) + y'(t_0)\Delta t$

$= $ ~~[illegible]~~ $= \boxed{y(t_0) + f(y_0)\Delta t}$

⇒ general : $y_{n+1} = y_n + f(y_n)\Delta t \quad \to$ First-order accurate

⇒ Δt must be small. but too small and round off errors will accumulate!

$\delta \simeq 10^{-16}$ for double precision

→ Problem w/ original → only estimates derivative at left end

2nd order method:

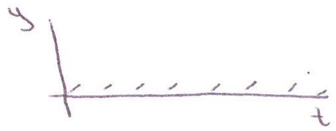
$$x_{n+1} = x_n + \frac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right]\Delta t \qquad \tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

Best way if you want to implement by hand → Runge-Kutta (4th order accurate)

⇒ DO AN EXAMPLE IN CLASS

In numerical methods, problem is to solve $\dot{y} = f(y, t)$ given that $y(t_0) = y_0$

⇒ $y(t_0) = y_0$ ↙ what is $y_0 +$ $\quad$ take step size Δt



$y_1 = y(t_1) = y(t_0 + \Delta t)$ ~~$y(t_1)$~~ $\to y(t_0$

$\swarrow \left(\frac{dy}{dt}\right)_{t_0}$ $\quad f(y(t_0))$

~~y_1~~ $\to t_0$ $\quad t_1 = t_0 + \Delta t$

$y(t_0 + \Delta t) = y(t_0) + \boxed{y'(t_0)}\Delta t$ $\quad y_1 = y(t_1) =$

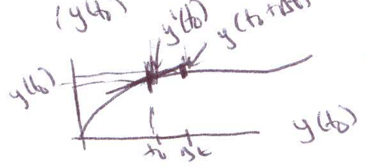
$y(t_1) = y(t_0) + f(y_0)\Delta t$ $\qquad y'(t_0) = \lim_{\Delta t \to 0} \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t}$ { eq. for der.

$y(t_2) = y(t_1) + y$

$y(t_2) = y(t_1 + \Delta t) = y(t_1) + y'(t_1)\Delta t$

$y_2 = y_1 + f(y_1)\Delta t$

$y_{n+1} = y_n + f(y_n)\Delta t$



$y(t_0 + \Delta t) = y(t_0) + y'(t_0)\Delta t$