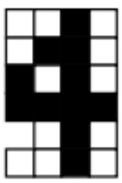
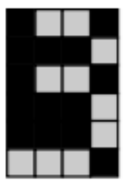
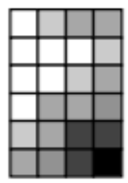
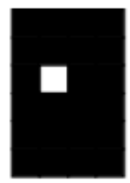
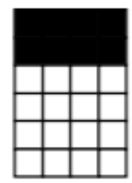
Problem Set – Linear Algebra I

Vectors

In class we talked about how vectors are just a list of numbers that can be interpreted as a direction or position in space. While this space can in some cases be literal physical space (such as the example with a mouse's x,y location), in general this space is called a **vector space**. A vector space can be abstract, like the space of visual stimuli or gene expression.

Let's imagine you have a very primitive black and white monitor. It only has a resolution of 4 pixels by 6 pixels. On this little monitor you can show pretty crappy images like the ones below:

The intensity or shade of each pixel is a value from 0 to 1. 0 for black, 0.5 would be medium grey, and 1 is white.

1. How could you describe these images as vectors? Describe in words what the idea is, and the pick one of the images and write it out as a vector. (Hint: Remember a vector is a *list* of numbers, not a table of numbers. Make up some rule that allows you to re-write each images as a list of numbers between 0 and 1.)

2. How many dimensions would this vector space have?

3. One of these images is a unit vector (has length 1), which image is it?

4. Pick another image and take the dot product between your image and this new image, using your vector representation. Remember the dot product is just a number – so your answer should just be one number.

5. Are any two of the images above not linearly independent of each other?

6. Consider all possible images. What kind of "volume" is this in the vector space you described? Is it like a

    a. line
    b. plane
    c. superman
    d. cube
    e. sphere
    f. triangle

Matrices

When we multiplied matrices and vectors, we saw that the matrix modified the vector by changing its direction and length. Multiplication by a matrix is called a linear transformation. Linear transformations can stretch, rotate, or flip vectors. In fact, every linear transformation can be **decomposed** into a rotation, a stretch and flip, and another rotation. In matrix multiplication, if A is our matrix representing a linear transformation, we'd say

$$A = UDV.$$

This is known as the Singular Value Decomposition, where we factorize A into several linear operations (specifically, a rotation (U), a stretch and flip (D), and another rotation (V)). We'll discuss how to do this, and why it's important (hint, this is what PCA is doing), in future lectures. But for now, let's keep our matrix multiplication abilities fresh.

What does it mean for U and V to just represent rotations? Rotation matrices are special matrices that can always be written as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

1. If $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, U is a rotation by 45 degrees, and V is a rotation by 90 degrees, what is A?