Political Science 100a/200a

Fall 2001

Sets[1]

1 Sets

Defn: A set is a collection of individuated or distinct objects or things. A member of a set is called an *element*.

1.1 Finite sets

Defn: A finite set has a finite number of elements. Finite sets are represented as follows:

$$S = \{Gore, Bush, Buchanan, Nader\}$$

- "S" is the name of the set. This is arbitrary. We could use any letter or symbol as the name. There are some conventions here, however. Sets are usually named with capital letters, and one usually picks a letter that has some mnemonic value. Here, "S" for "set," but we might have used, say, "C" for "candidates."
- "*Gore*," "*Bush*," etc. are elements of S. They are enclosed by the brackets "{" and "}" which are used to denote a set. The order of elements in the set does not matter. We could just as well write $S = \{Nader, Buchanan, Gore, Bush\}$. In general, two sets are the same if and only if they have exactly the same members.
- "$Gore \in S$" reads "*Gore* is a member of the set S." "$\in$" means "is a member of" or "is in".
- The members of sets must all be *distinct* objects. For example A = {*Bush*, *Gore*, *Bush*} is not a set if both *Bush*s refer to the same object.

  Note, however, that you may encounter sets such as $S = \{cooperate, cooperate\}$. Here it would be implicitly understood that the elements refer to (in this example) a strategy for player 1 (the first "cooperate") and a strategy for player 2 (the second one).
- The elements of sets need not be numbers. They can be practically any things that can be named.

[1] Notes by James D. Fearon, Dept. of Political Science, Stanford University, October 2001.

1.2 Infinite sets

An *infinite set* is a set with an infinite number of elements. For example, the set S that is the set of all numbers between 0 and 1 is an infinite set, and so is the set X that is the set of all actions I could take in the next instant.

Obviously, we can't use the previous method (called "enumeration") for representing an infinite set. Instead, we typically use the following method:

$$S = \{s : 0 < s < 1\}, \text{ and}$$

$$X = \{x : x \text{ is an action I could take in the next instant}\}$$

.

Here, ":" reads "such that."

In this method, a set is defined by specifying the property or properties that characterizes or is true of all elements of the set.

Occasionally, you will also see notation like the following for an infinite set:

$$T = \{0, 1, 2, 3, \ldots\}$$

where here it is understood that the ellipsis means that the set proceeds as indicated.

1.3 Relations between sets

a. *inclusion*: $A \subset B$ if $\forall a \in A$, $a \in B$. The $\forall$ symbol reads "for all" or "for every." A set A such that $A \subset B$ is called a *subset* of B. Show Venn Diagram.

   For example, if the set $T = \{Bush, Gore\}$, then $T \subset S$ defined above ($S = \{Bush, Buchanan, Gore\}$).

   Note that $A \subset A$ for any set A.

b. *identity*: $A = B$ if $A \subset B$ and $B \subset A$.

c. *disjoint sets*: A and B are *disjoint* if they have no elements in common. That is, if there does not exist an $a \in A$ such that $a \in B$. Show Venn diagram.

We should also define a special set called the *null set* or the *empty set*, which is a set that has no elements. This is denoted $\emptyset$.

Note that by convention the null set is a subset of any set, that is, $\emptyset \subset A$ for any set A.

Digression on subsets:

- The number of elements in a set A is often written as $|A|$.
- The total number of subsets of a set A is $2^{|A|}$ (including the null set as a subset).
- The set of all subsets of a set A is usually called *the power set* of A, and is denoted 2^A.

1.4 Operations on sets

- The *union* of two sets is the set formed by the combination of two sets. Formally,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

  Show Venn diagram.

- The *intersection* of two sets A and B is the set of elements that are in both A and B (i.e., the elements they have in common). Formally,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

  Show Venn diagram.

  Thus, sets A and B are disjoint if $A \cap B = \emptyset$.

- Suppose that $A \subset S$. Then *the complement* of A, denoted A^c or $\bar{A}$, is the set of elements that are members of S but not members of A. Formally,

$$A^c = \{x : x \notin A\},$$

  where it is understood that S is the "universal" set with respect to which the complement of A has meaning.

  Show Venn diagram.

- The notation $A \backslash \{a\}$ means the subset of A that is left over when the subset $\{a\}$ is removed.

One last operation on sets, a very important and conceptually more difficult one: The Cartesian product.

- The *Cartesian product* of two sets A and B is denoted $A \times B$ and is defined as the set composed of all ordered pairs that can be formed by taking an element from A and an element from B.

This isn't such a great definition as it stands because I haven't told you what an ordered pair is. An ordered pair is an instance of a type of set called an ordered set, which is a set where the order in which the elements are listed matters.

In social science problems, we often identify the things being analyzed by giving lists of characteristics. For example, a voter in a survey might be identified with a list of characteristics like the following: (age, sex, party id, income).

Clearly order matters here if we are to know what refers to what.

We call such lists "ordered sets" or "ordered n-tuples," or in the case of two elements, an ordered pair.

Recall that $\{a, b\}$ and $\{b, a\}$ refer to the same set. By contrast, to denote an ordered pair we write (a, b), and it is not the case that (b, a) is the same thing as (a, b), unless it happens that $b = a$.

Now, back to Cartesian products. Suppose $A = \{10k, 20k, 30k\}$ is a set of possible incomes, and $B = \{20s, 30s, 40s\}$ is a set of ages.

Then the Cartesian product $A \times B$ is the set of ordered pairs formed by taking an element from A and an element from B in every way that is possible. Thus

$$A \times B = \{(10k, 20s), (10k, 30s), (10k, 40s), (20k, 20s), (20k, 30s), \ldots\}.$$

(How many elements are there, total, in this set?)

We will see more examples of the Cartesian product when I talk about numbers, and this operation is used constantly in game theory notation. (The reason is that in game theory we are interested in choices from sets of options, so that a "strategy" is often expressed as the cartesian product of several sets of discrete options.)

Things to note about cartesian products:

- You can form the Cartesian product of *any* sets, no matter how diverse the nature of their elements. For example, let $A = \{x : 0 < x < 1\}$ and $B = \{cooperate, defect\}$.

  What is $A \times B$? Or, ask what is a typical element of $A \times B$?

A typical element is an ordered pair $(x, cooperate)$ or $(x, defect)$, where x is a number between 0 and 1. Ok?

- Order matters. If $A \neq B$, then it is not true that $A \times B = B \times A$, since order in each pair matters.
- We can take the cartesian product of more than two sets, e.g., $A \times B \times C$. This is the set composed of ordered triples (a, b, c), in all the ways you can form these.