# Psych 253
# Advanced Statistical Modeling

## Review of Differential Calculus

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# Definition of Derivative

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(cancel leading term)

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$$= \lim_{\Delta x \to 0} \frac{2\Delta x \cdot x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = \boxed{2x}$$

(cancel leading term) (divide out **Δx**) (take limit)

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$$= \lim_{\Delta x \to 0} \frac{\sum_{i=0}^{n} \binom{n}{i} x^{n-i} (\Delta x)^i - x^n}{\Delta x} \quad \text{(binomial thm.)}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + n x^{n-1} \cdot (\Delta x) + \sum_{i=2}^{n} \binom{n}{i} x^{n-i} (\Delta x)^i - x^n}{\Delta x}$$

$$= n x^{n-1} + \lim_{\Delta x \to 0} \sum_{i=2}^{n} \binom{n}{i} x^{n-i} (\Delta x)^{i-1} = \boxed{n x^{n-1}}$$

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NB: these three steps, in this specific computation, represent Newton's most important mathematical contributions, shifting the course of history after ever

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# Interpretation as Local Slope

***F'(x)*** is the slope of the line that is tangent to the graph of ***F(x)*** at ***x***:

Suppose that line is:

$$y = mx + b$$

Then:

$$F(x) = F'(x)x + b$$

So:

$$b = F(x) - F'(x)x$$

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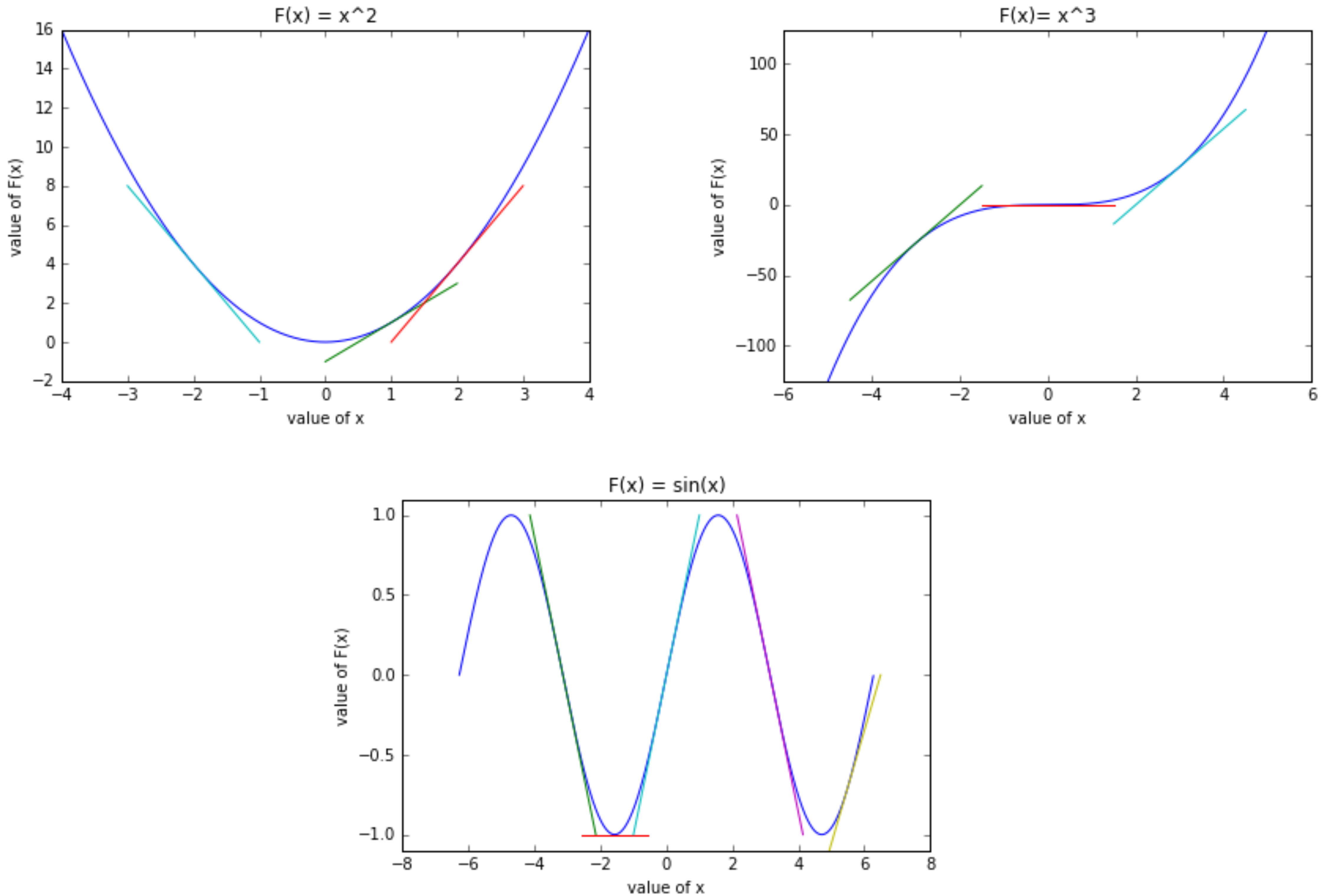


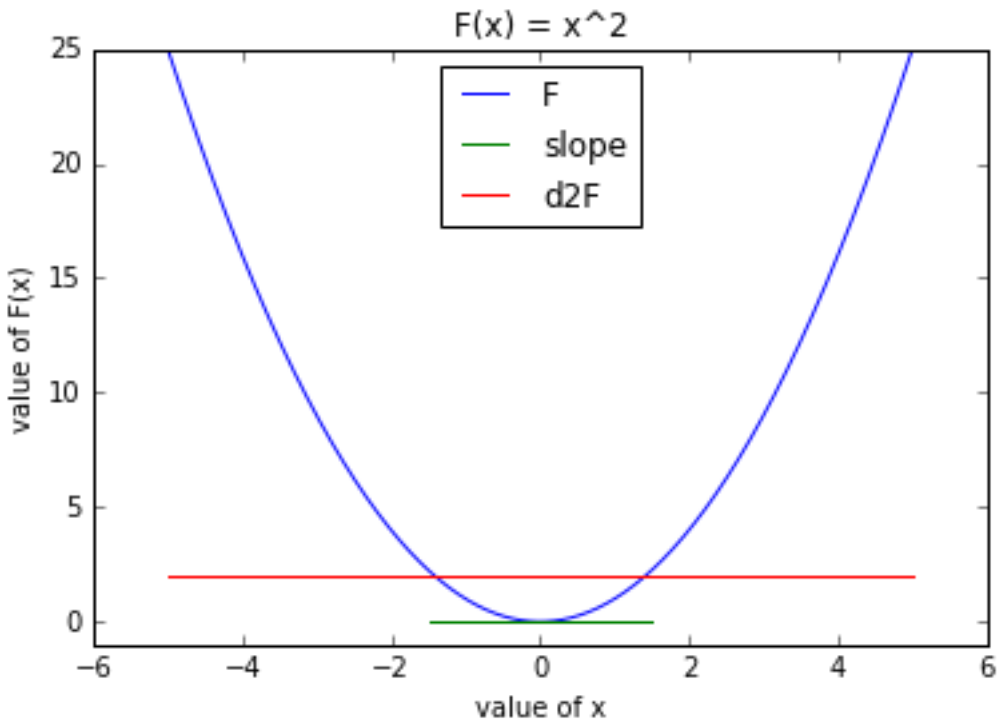
# Higher Derivatives

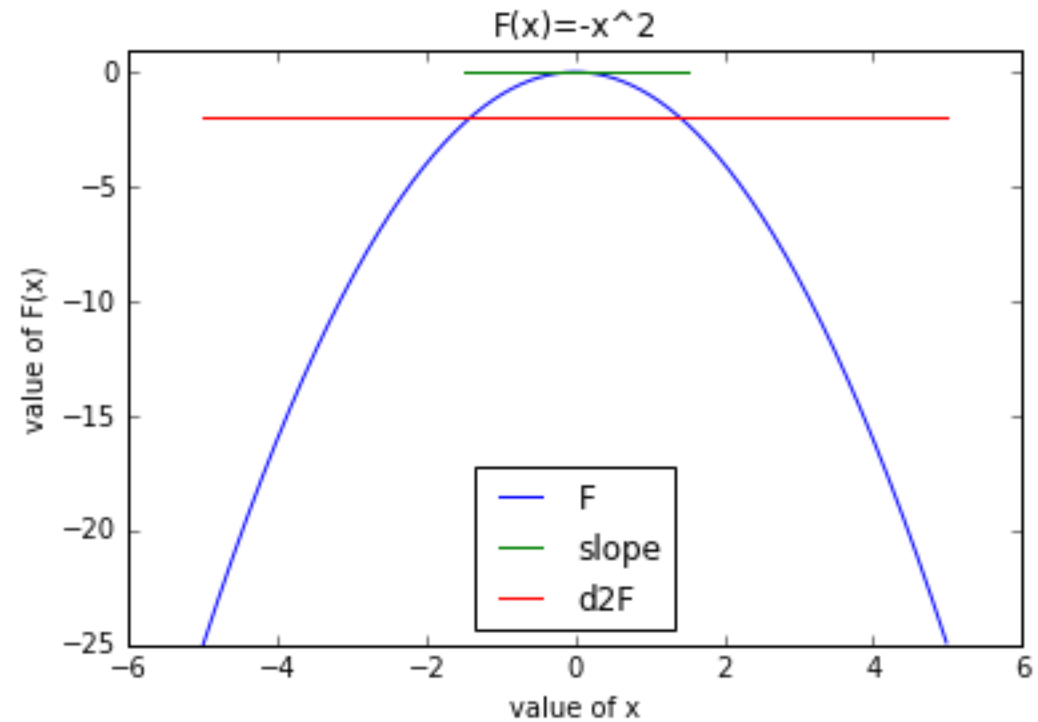
The nth derivative is the derivative of the (n-1)st derivative:

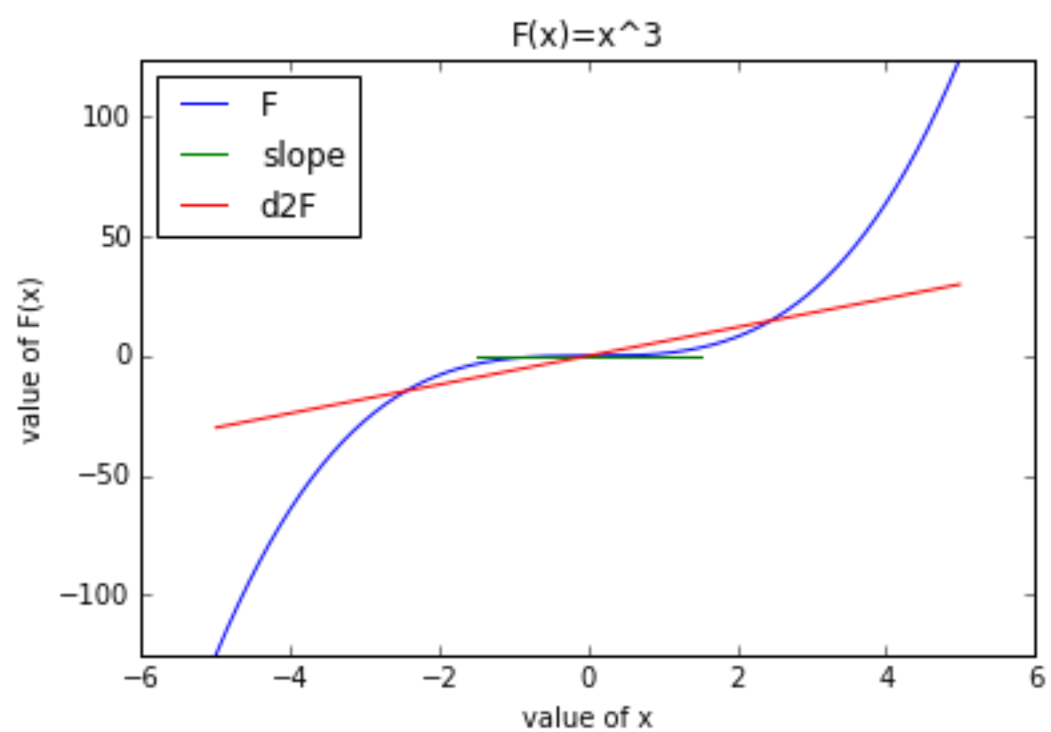
$$F^{(n)}(x) = \frac{d^n F}{dx^n} = \frac{d}{dx}(F^{(n-1)}(x))$$

# Higher Derivatives

E.g., second derivatives:



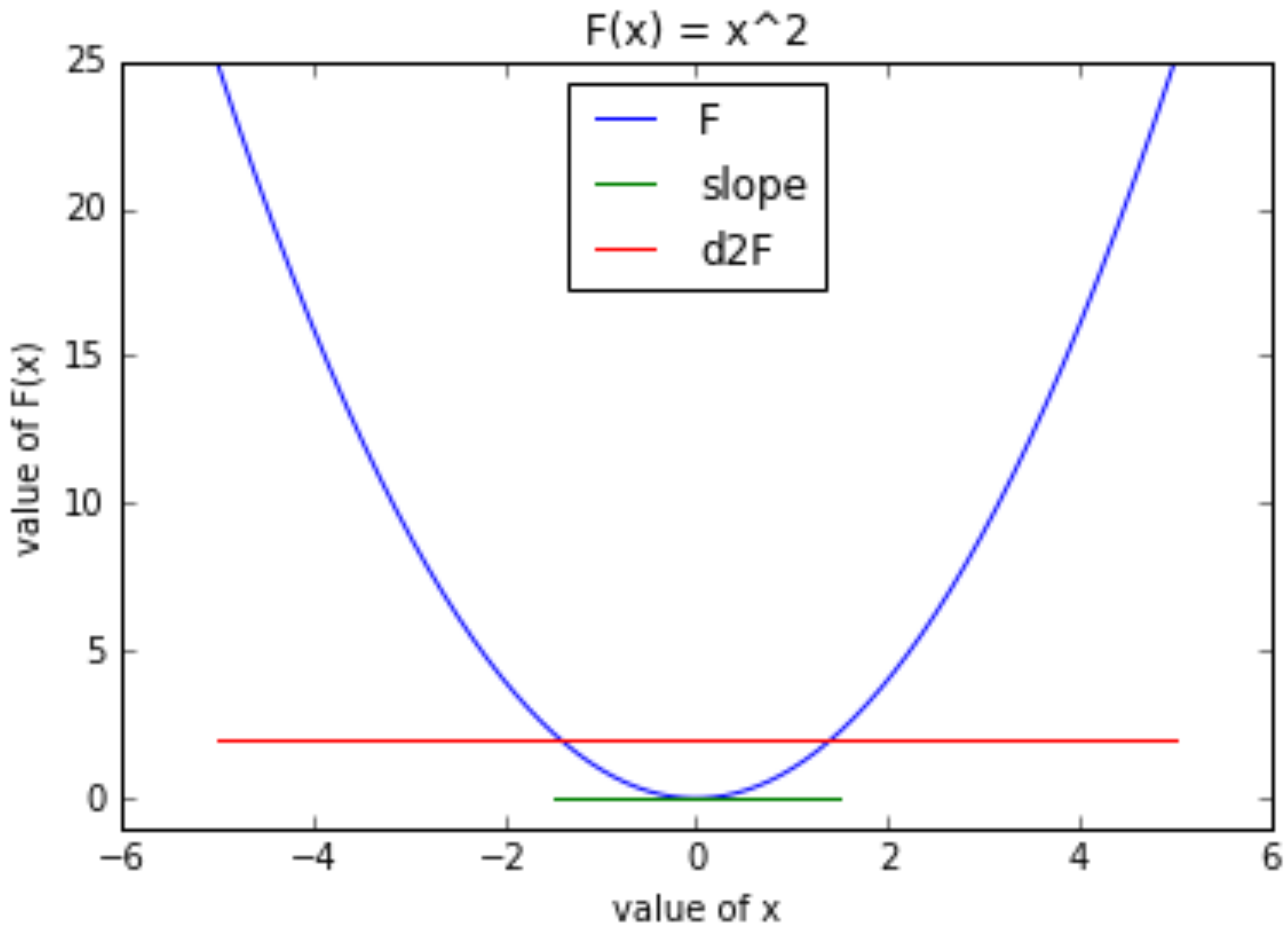



# Critical Points and Local Optima

Critical points are those ***x**** such that:

$$\left.\frac{dF}{dx}\right|_{x=x^*} = 0$$



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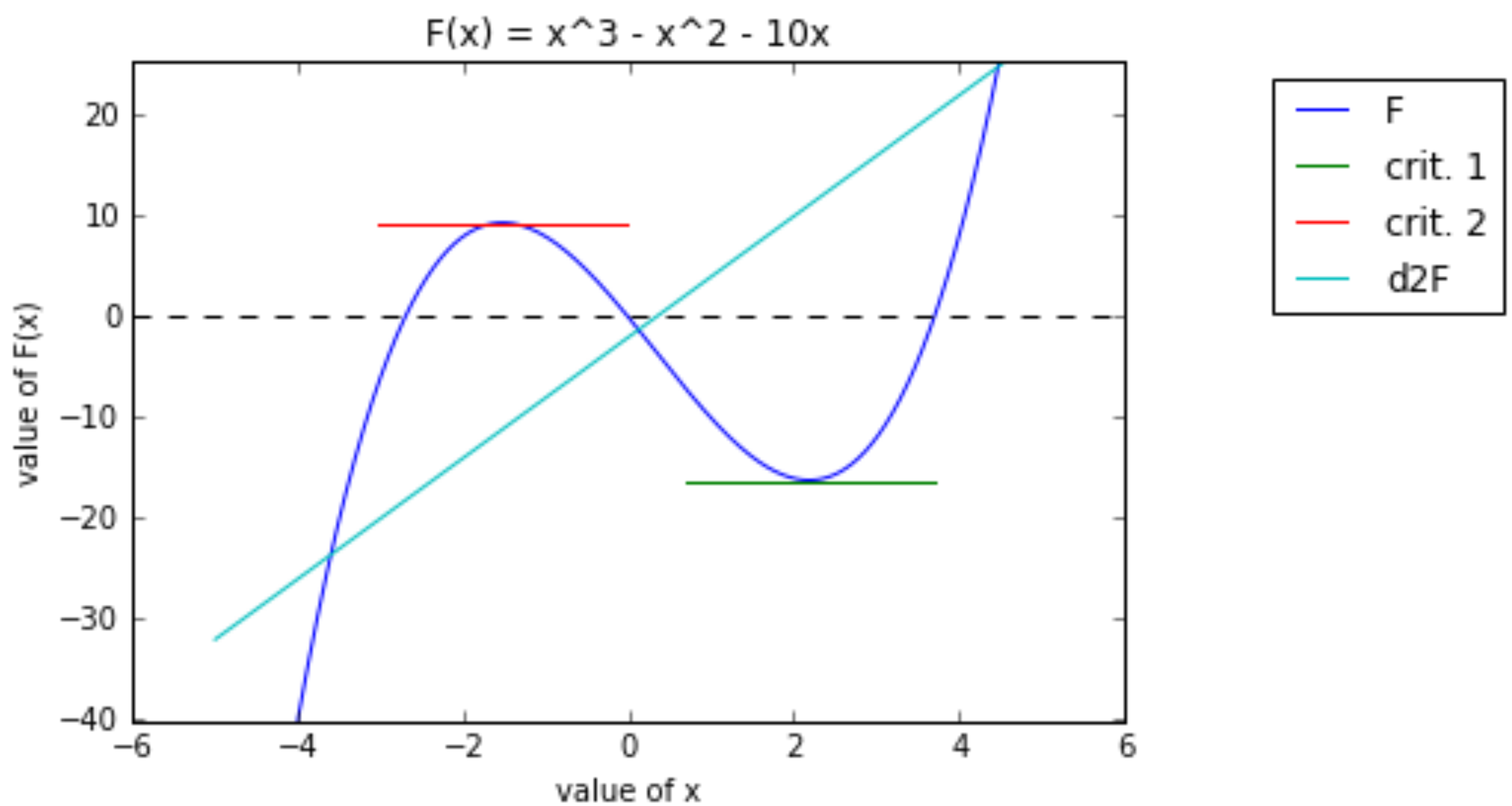

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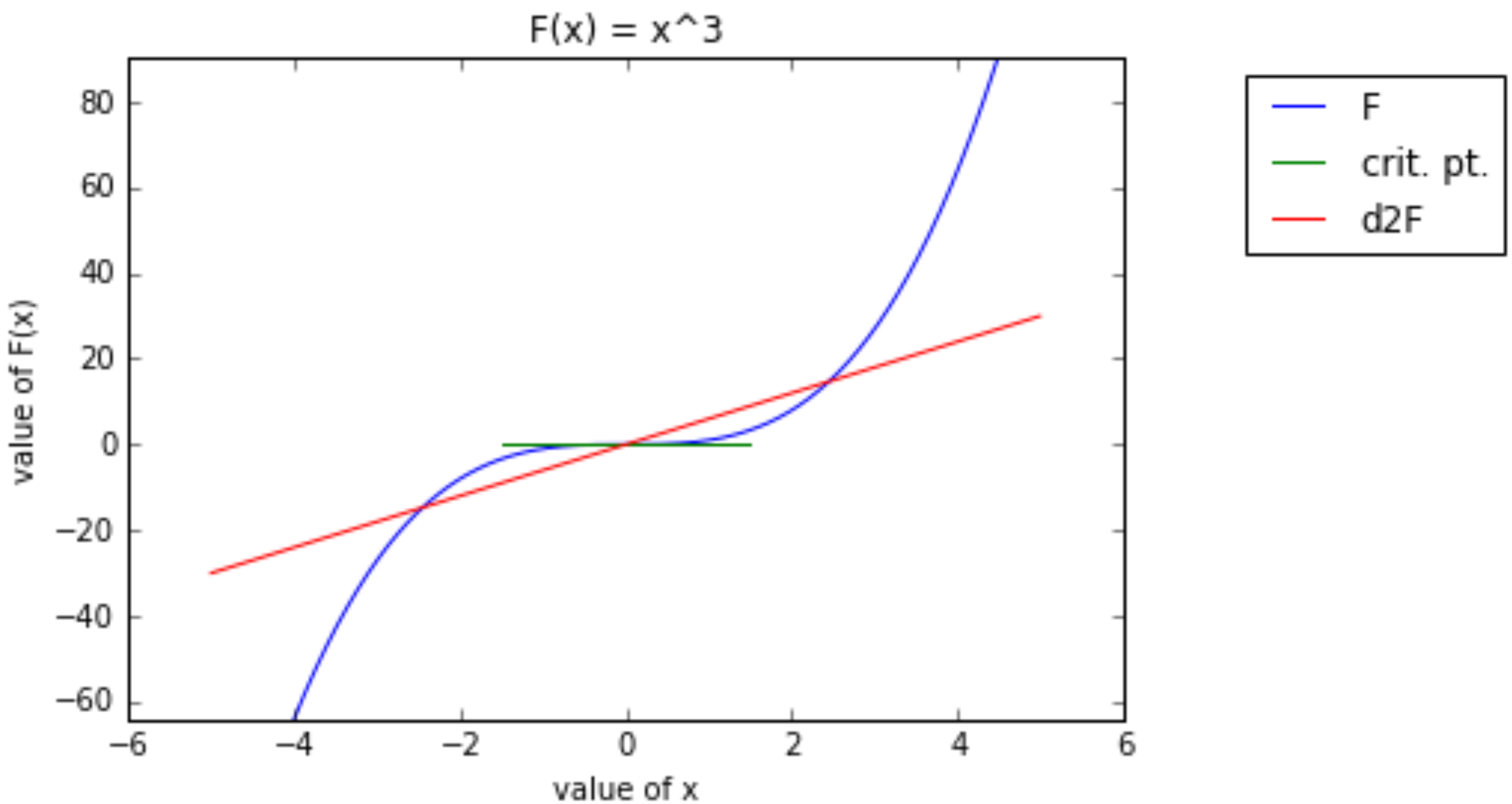

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Second derivative test:

$$\text{critical point } x^* = \left\{ \begin{array}{ll} \text{local maximum,} & \text{if } d^2F/dx^2 < 0 \\ \text{local minimum,} & \text{if } d^2F/dx^2 > 0 \\ \text{saddle point (maybe),} & \text{if } d^2F/dx^2 = 0 \end{array} \right\}$$

# Taylor Series

Let's say we wanted to approximate a function ***F*** around a point ***x0*** as a series in **Δ*x***:

$$F(x_0 + \Delta x) = a_0 + a_1 \Delta x + a_2 [\Delta x]^2 + a_3 [\Delta x]^3 + \ldots$$

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constant term

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Let's say we wanted to approximate a function ***F*** around a point ***x0*** as a series in **Δx**:

$$F(x_0 + \Delta x) = \overbrace{\boxed{a_0}}^{\text{constant term}} + \underbrace{\boxed{a_1 \Delta x}}_{\text{linear team}} + \overbrace{\boxed{a_2 [\Delta x]^2}}^{\text{quadratic term}} + \underbrace{\boxed{a_3 [\Delta x]^3}}_{\text{cubic term}} + \dots$$

$$= \sum_{i=0}^{\infty} a_i [\Delta x]^i$$

First of all: $\Delta x = 0 \rightarrow a_0 = F(x_0)$

Then: $\dfrac{dF}{d\Delta x}(x_0 + \Delta x) = a_1 + 2a_2 \Delta x + \dots = \sum_{i=1}^{\infty} i \cdot a_i [\Delta x]^{i-1}$

So: $\Delta x = 0 \rightarrow a_1 = \left.\dfrac{dF}{dx}\right|_{x=x_0}$

# Taylor Series

But if $$\frac{dF}{d\Delta x}(x_0 + \Delta x) = a_1 + 2a_2\Delta x + \ldots = \sum_{i=1}^{\infty} i \cdot a_i [\Delta x]^{i-1}$$

then $$\frac{d^2F}{d\Delta x^2}(x_0 + \Delta x) = 2a_2 + 6a_3\Delta x + \ldots = \sum_{i=2}^{\infty} i \cdot (i-1) \cdot a_i [\Delta x]^{i-2}$$

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In general:

$$k \cdot (k-1) \ldots 2 \cdot 1 \cdot a_k = \frac{d^kF}{dx^k}\bigg|_{x=x_0}$$

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In general:

$$k \cdot (k-1) \ldots 2 \cdot 1 \cdot a_k = \frac{d^kF}{dx^k}\bigg|_{x=x_0}$$

Or:

$$a_k = \frac{1}{k!}\frac{d^kF}{dx^k}\bigg|_{x=x_0}$$

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$$= \sum_{k=0}^{\infty} a_k [\Delta x]^k$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{d^k F}{dx^k} \right|_{x=x_0} \cdot [\Delta x]^k}$$

Taylor Series Approximation

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So now we need to calculate these two limits

# Taylor Series

We'll use the Taylor Series concept to calculate the first limit.

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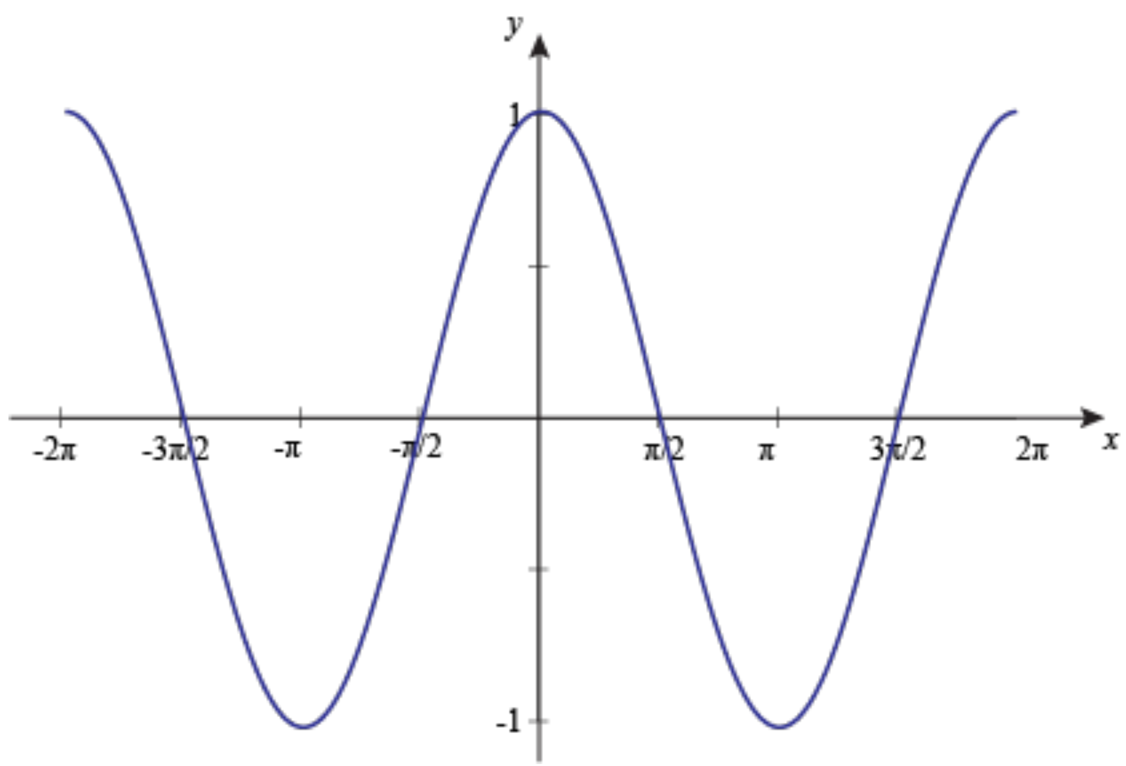
Now, ***cos(x)*** is an even function around 0:




$$\cos(x) = \cos(-x)$$

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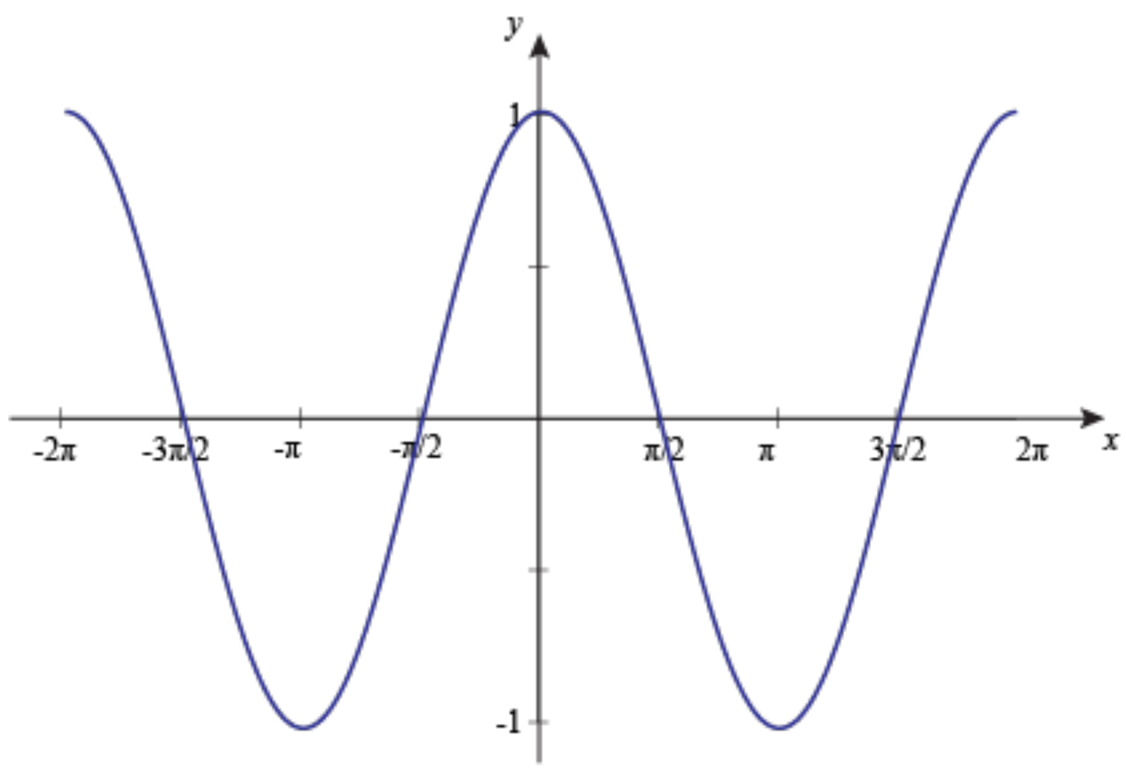


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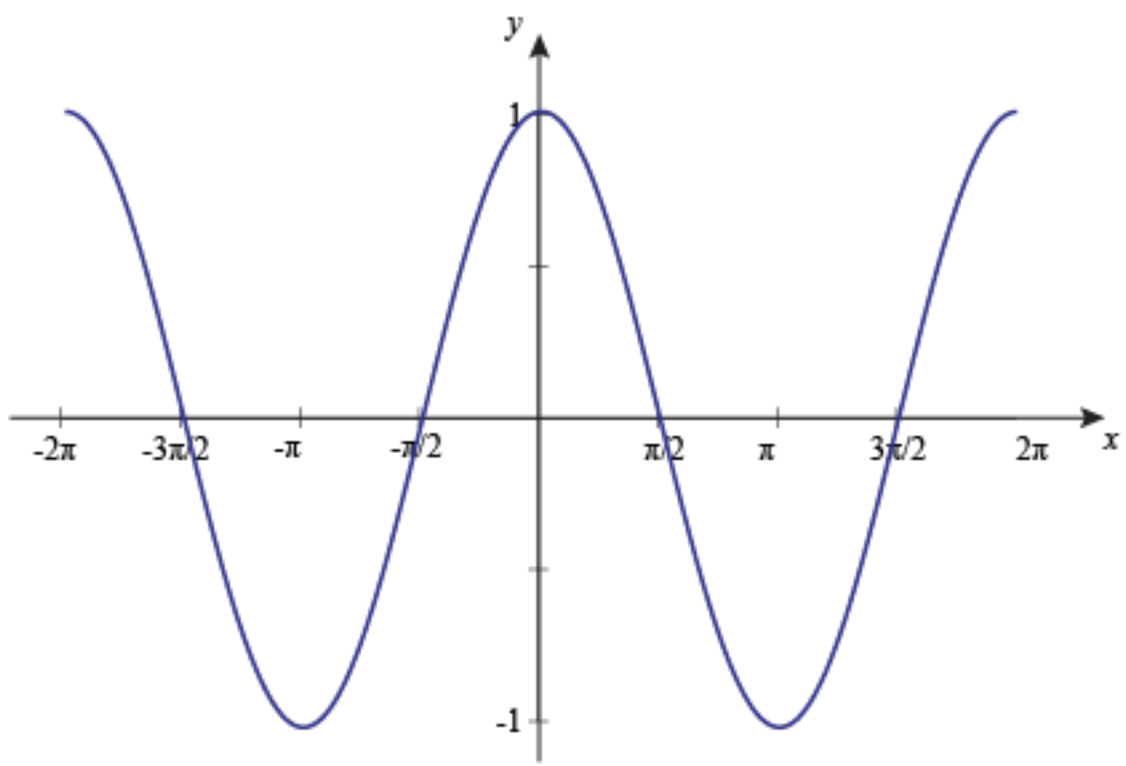


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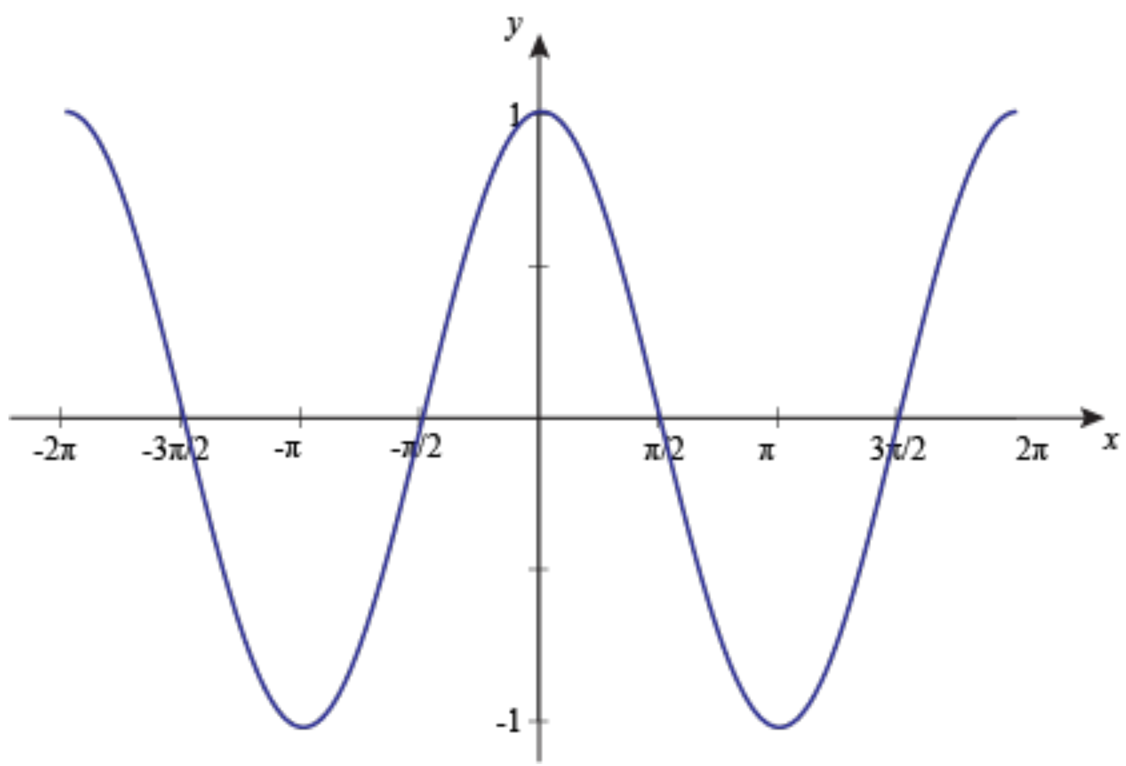


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So cosine's Taylor series is:

$$\cos(\Delta x) = 1 + a_2 \cos''(0)(\Delta x)^2 + \dots$$

# Taylor Series

Since

$$\cos(\Delta x) = 1 + a_2 \cos''(0)(\Delta x)^2 + \ldots$$

we have:

$$\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{a_2 \cos''(0)(\Delta x)^2}{\Delta x}$$

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So

$$\sin'(x) = \sin(x) \cancelto{= 0}{\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x}} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

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Since

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Observation: note that we didn't need to know cosine's full taylor series for this calculation — just that cosine's being an even fn meant that it's first order term disappeared … this is one reason ***why*** Taylor series are useful

we have:

$$\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{a}{\phantom{x}}$$

$$= \lim_{\Delta x \to 0} a_2$$

$$= 0$$

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So

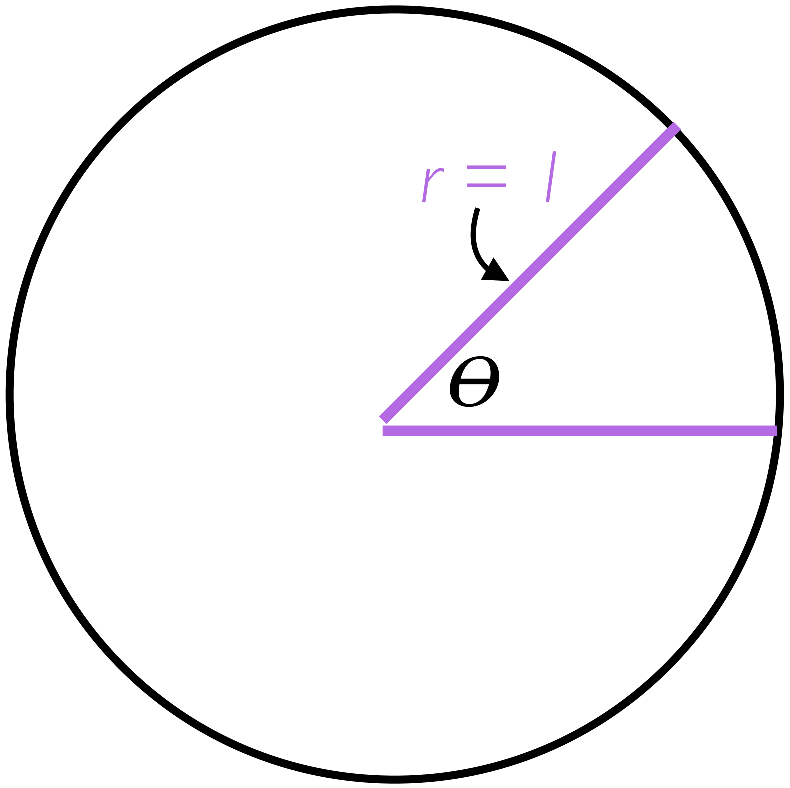
$$\sin'(x) = \sin(x) \underbrace{\cancel{\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x}}}_{= 0} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$$

$$= \cos(x) \underbrace{\boxed{\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}}}_{= \sin'(0)}$$

Ok, but what is ***this*** limit?

# Taylor Series

Assume circle is radius 1



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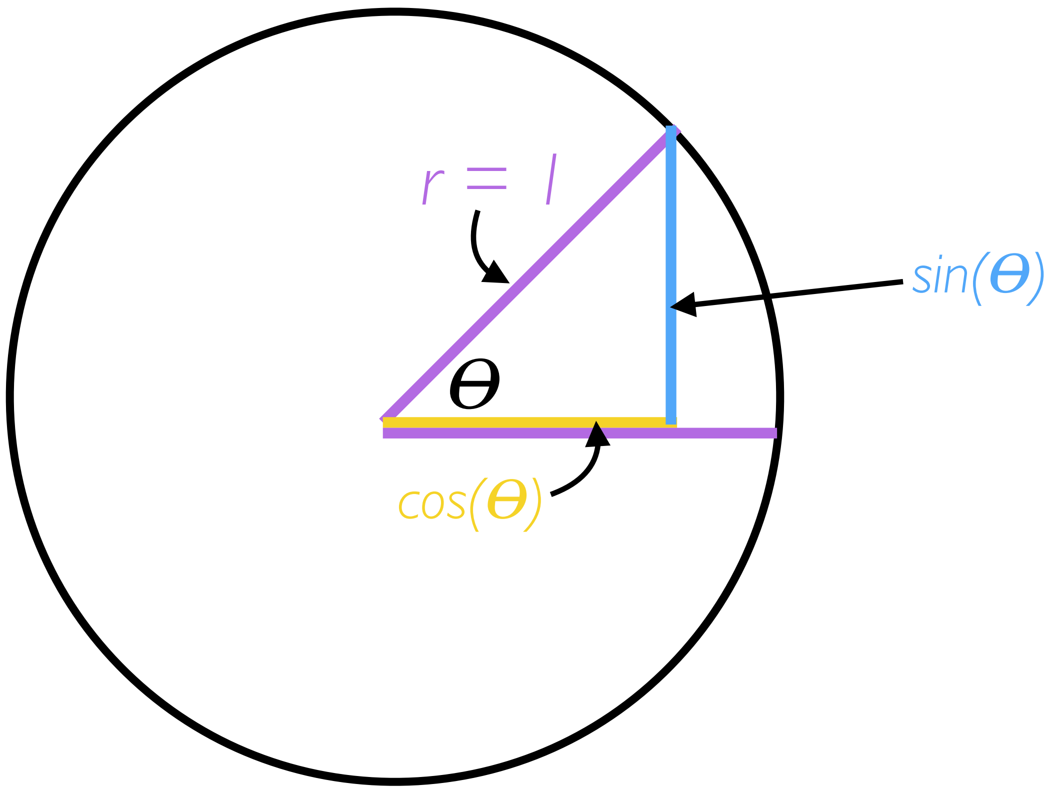

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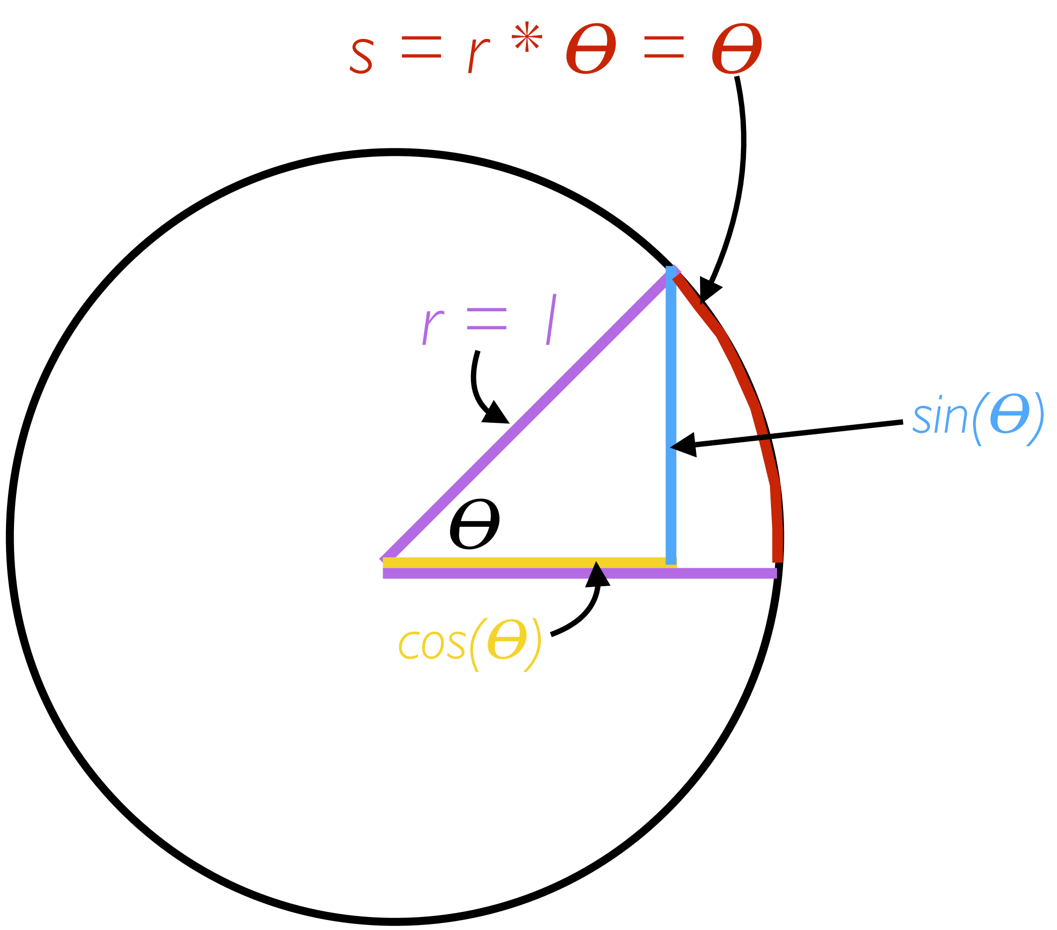

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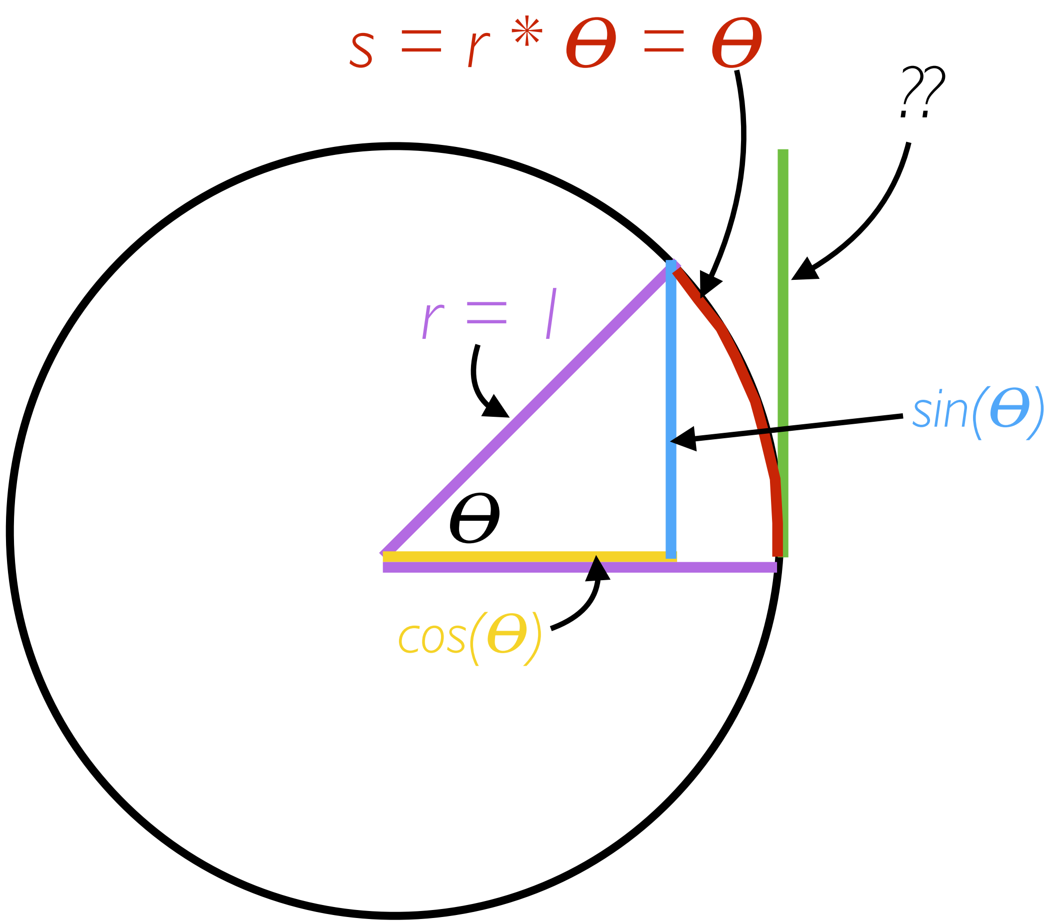

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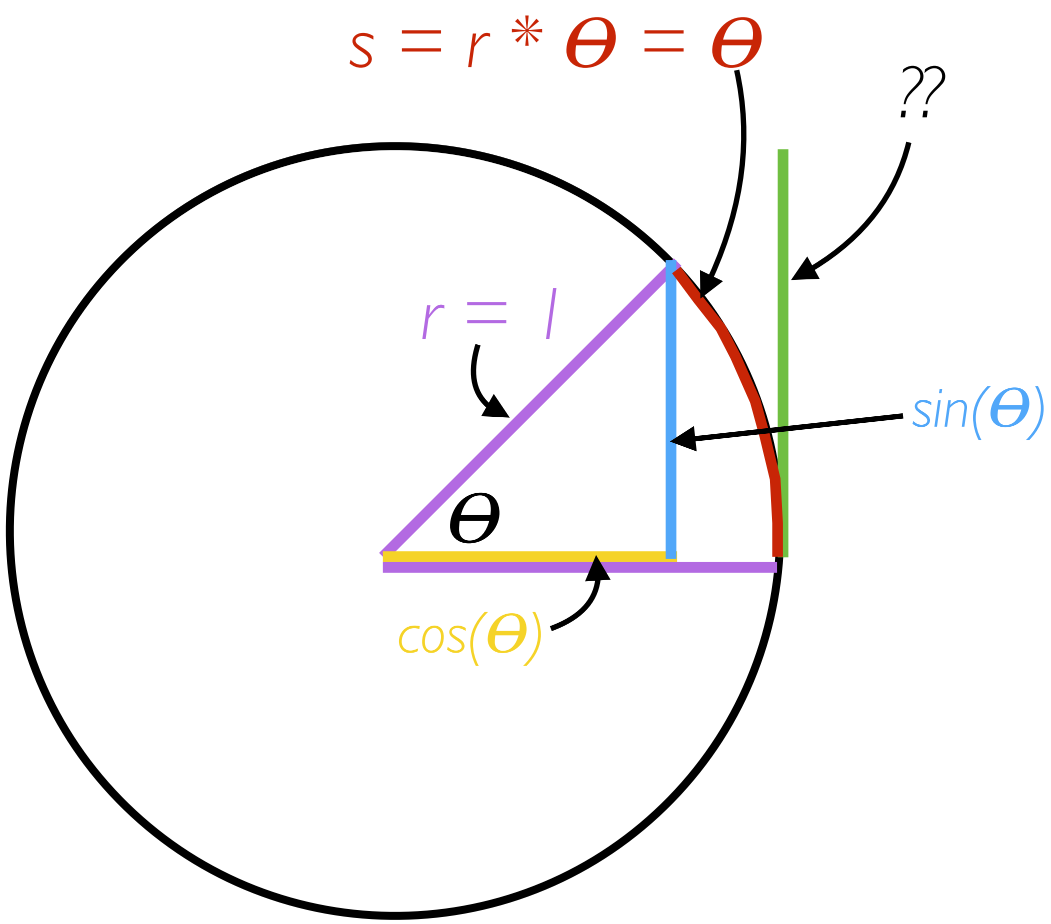


$$\frac{\sin(\theta)}{\cos(\theta)}$$ = ratio of blue to yellow = ratio of green to **r** = green

$$=$$

$$\tan(\theta)$$

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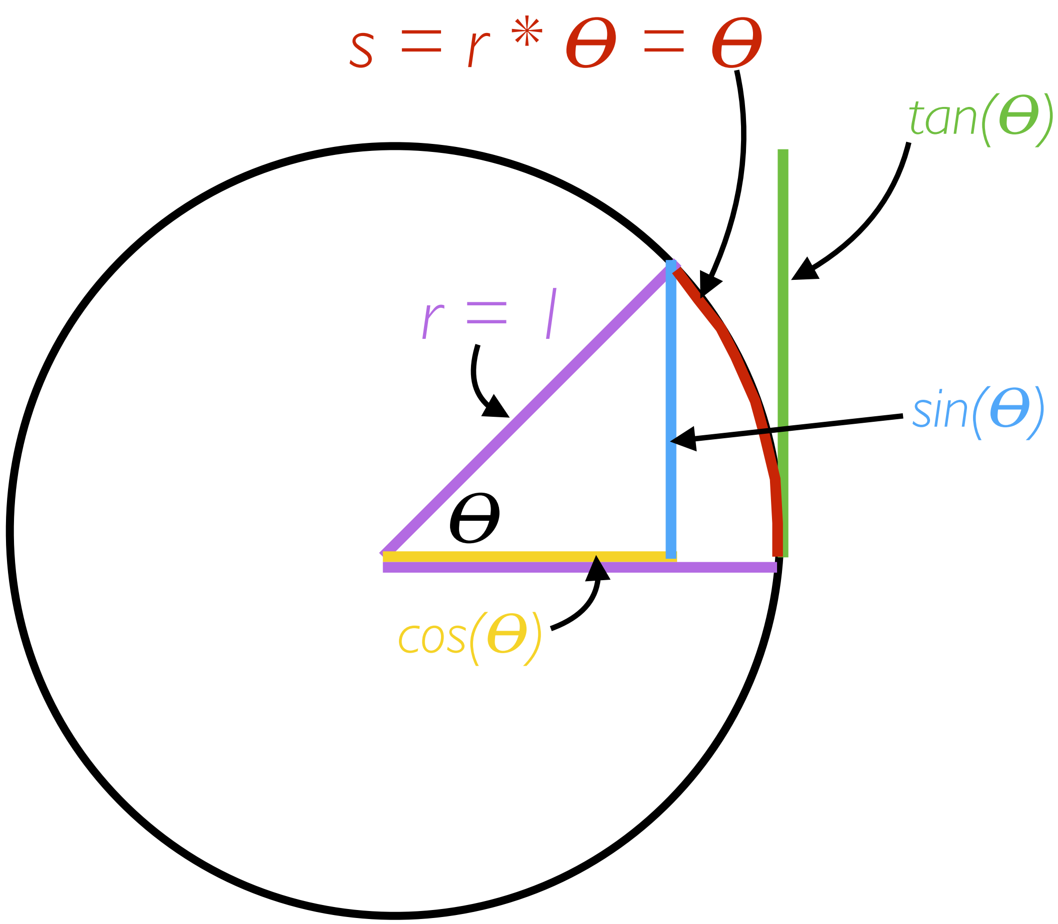

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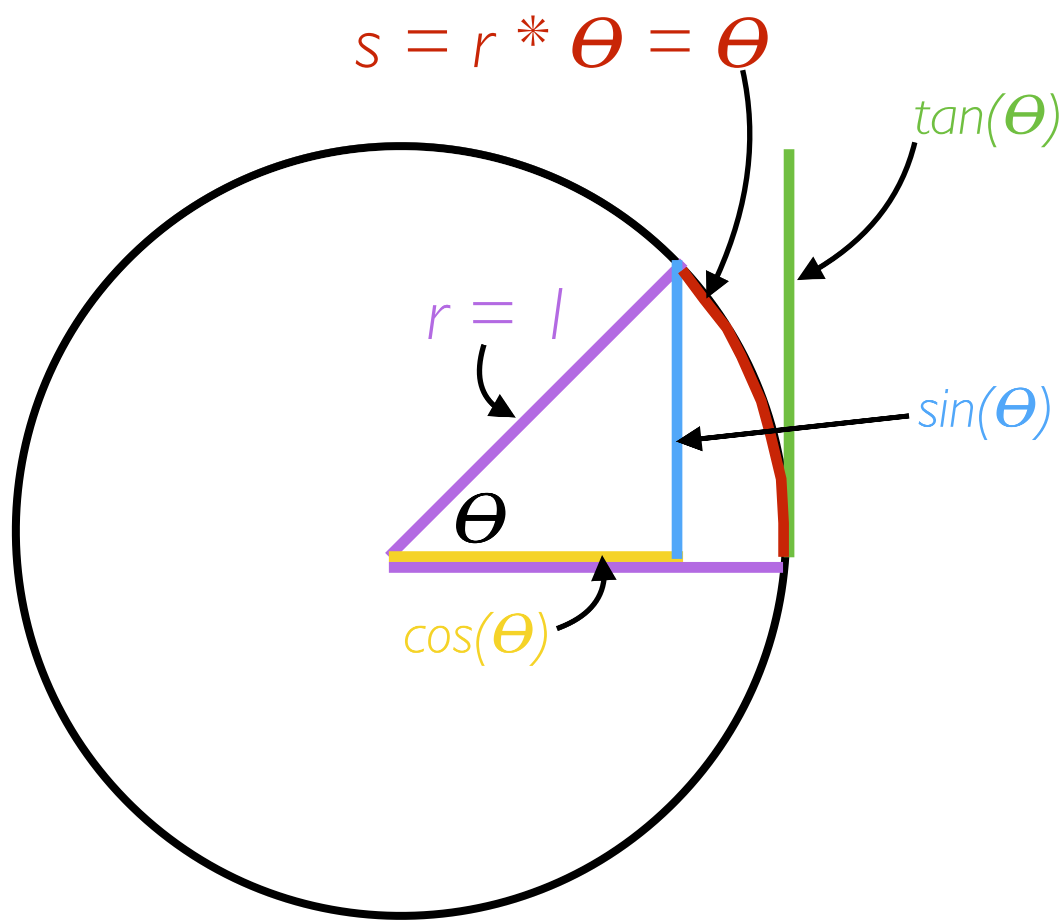


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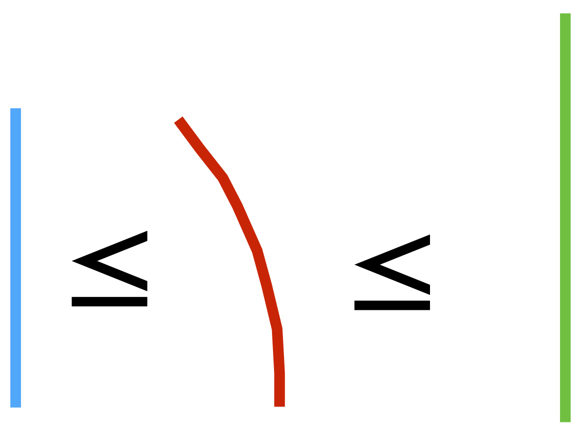

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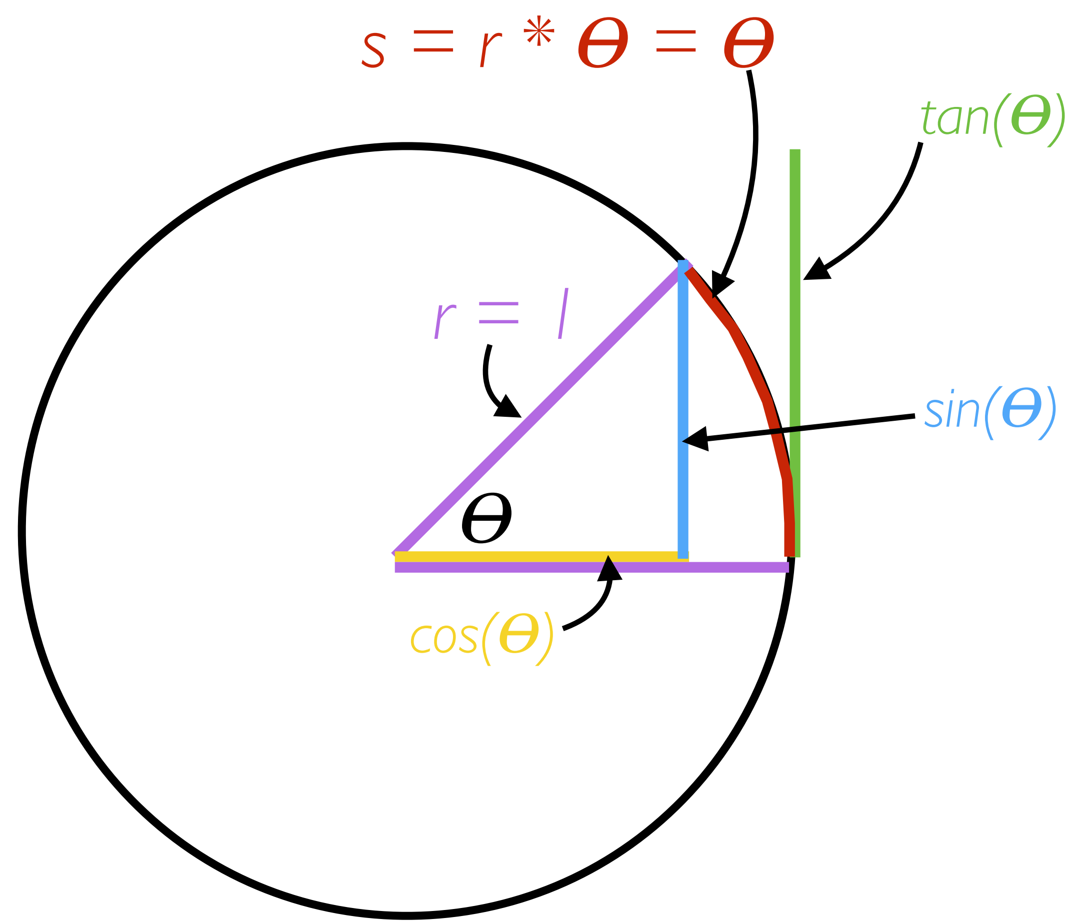


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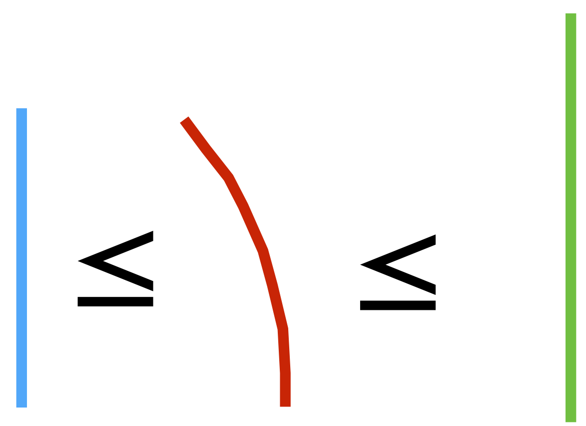


So:

$$\sin(\theta) \leq \theta \leq \tan(\theta)$$

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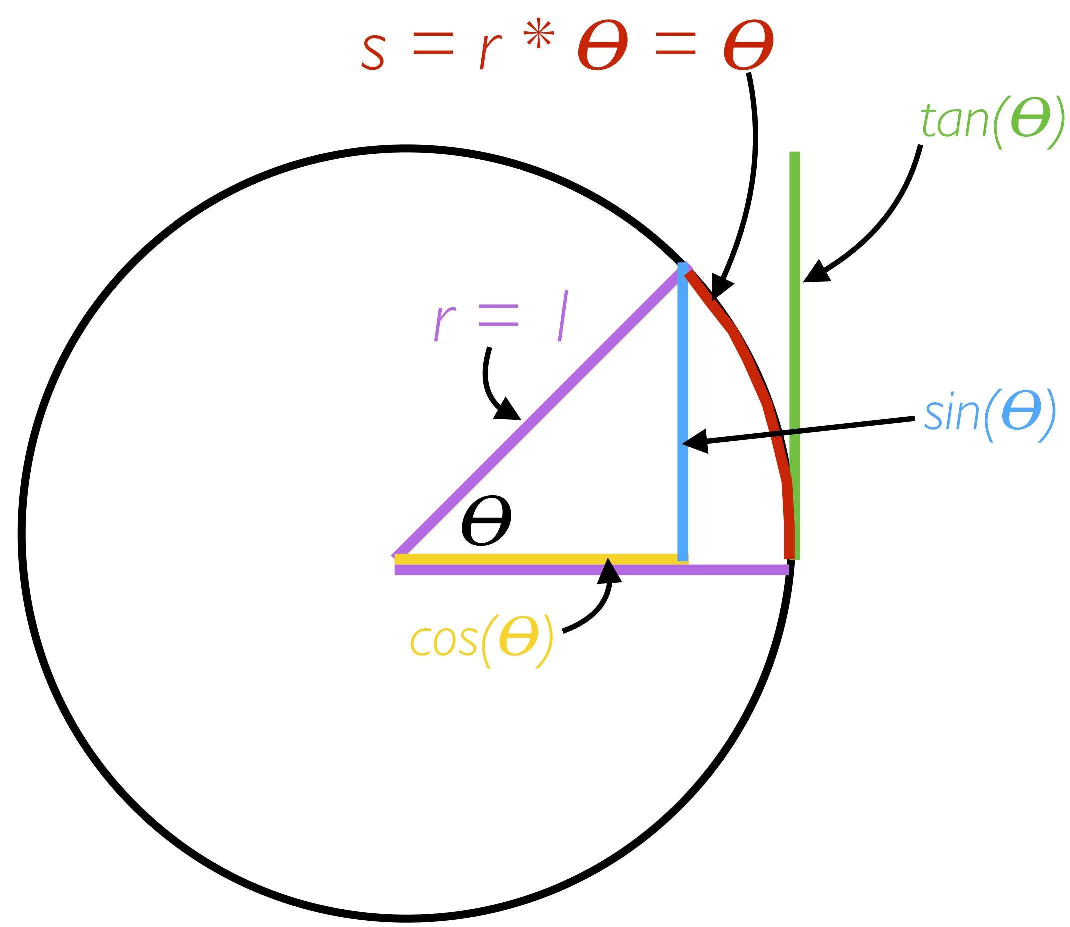


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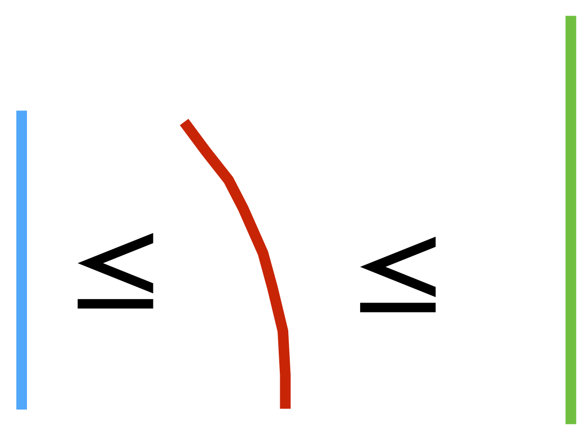


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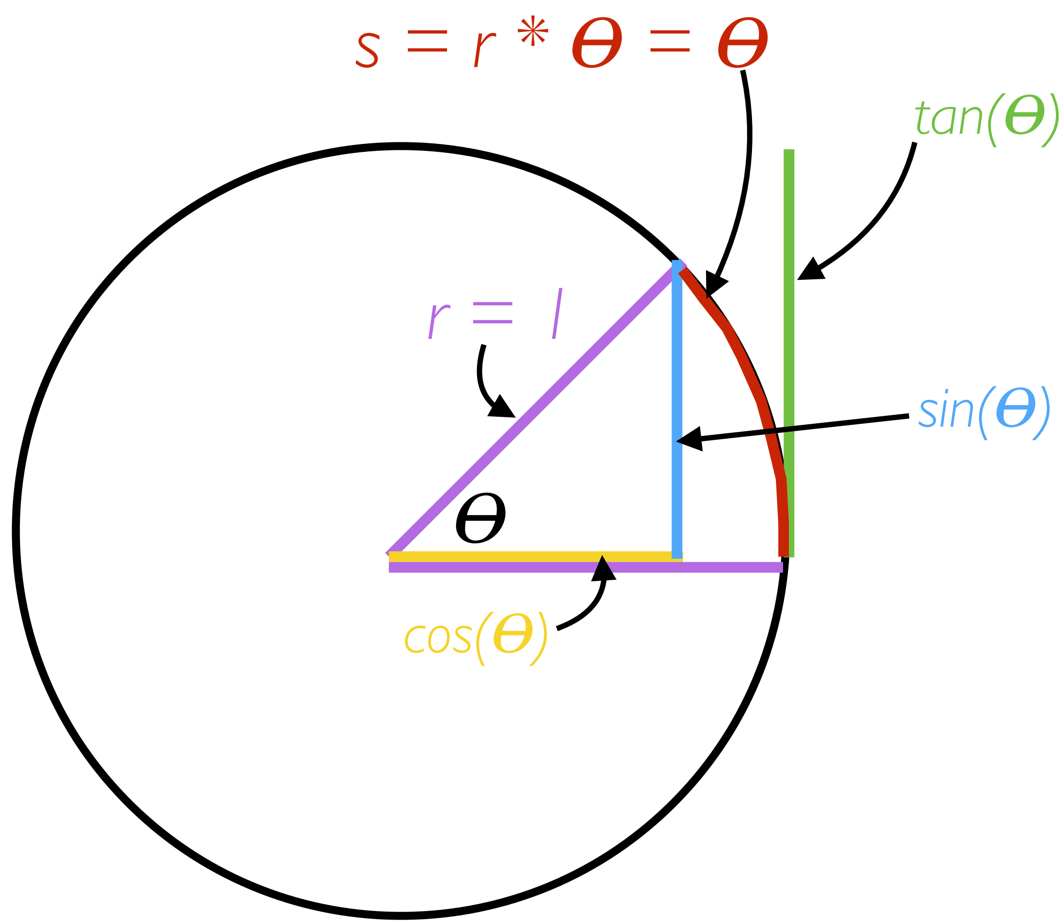


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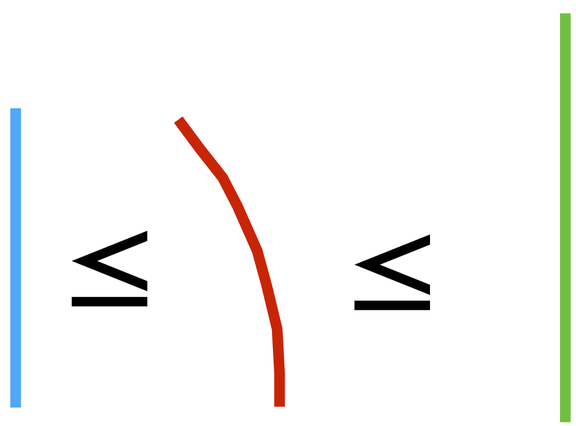


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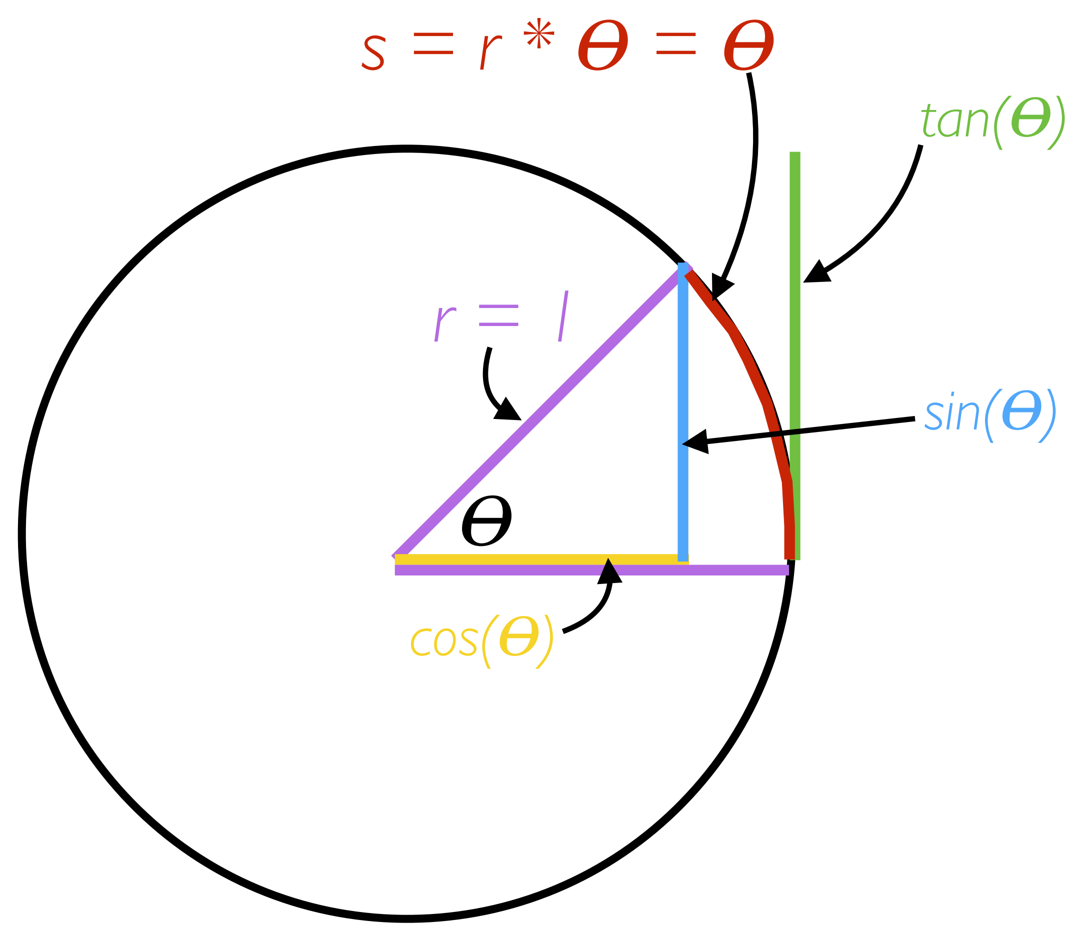


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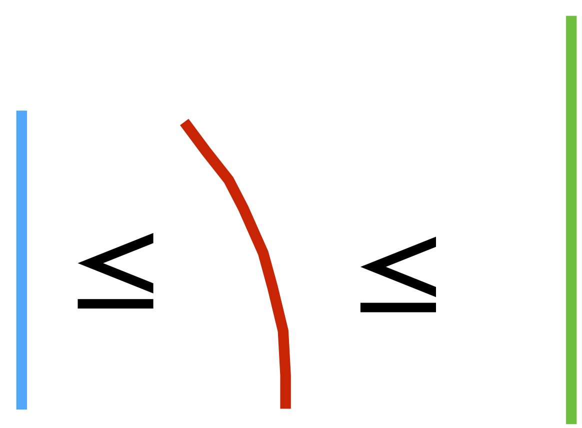


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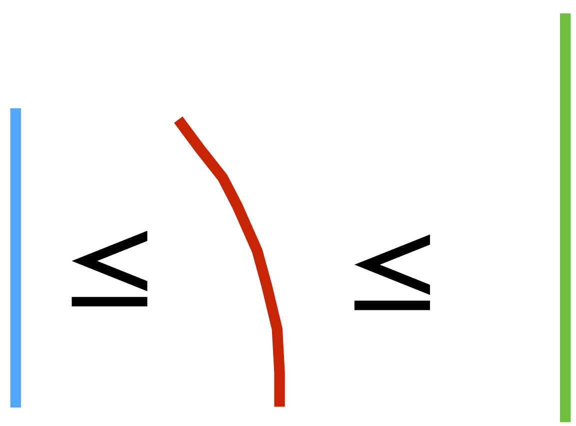
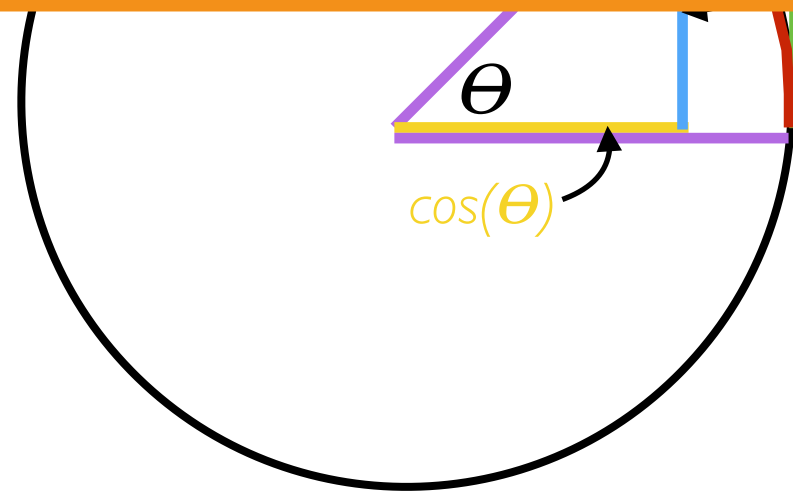
BEAUTIFUL!!!

# Taylor Series

[Tangential Observation: this calculation of this ***sin(x)/x*** limit much "more specific" mathematically than the ***(cos(x) - 1) / x*** limit, because the latter just depended on ***cos(x)*** being an even function — any even function would have had a similar Taylor series. However, to get ***sin(x)/x*** right, we really needed to understand the specifics of trigonometry.]



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Anyway, we finally have:

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Similarly we can show:

$$\sin''(x) = \cos'(x) = -\sin(x)$$

Thus:

$$\sin^{(3)}(x) = -\cos(x)$$

And:

$$\sin^{(4)}(x) = \sin(x)$$

# Taylor Series

With that in mind we can calculate full Taylor series' for sine and cosine (around 0):

$$\sin(x) = \sin(0) + \sin'(0)x + \frac{1}{2}\sin^{(2)}(0)x^2 + \ldots$$

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And similarly: $$\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}$$

# Chain Rule

Assume we have a function that's a composition of functions:

$$h(x) = g(f(x))$$

Then the chain rule says:

$$\frac{dh}{dx}(x) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x) = g'(f(x)) \cdot f'(x)$$

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Since by definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

we have:

$$f(x+\Delta x) \sim f'(x)\Delta x + f(x)$$

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Now apply the same reasoning to **g** and treating ***f'(x)Δx*** as the "new **Δ*x***" and f(x) as the "new ***x***":

$$g(f'(x)\Delta x + f(x)) \sim g'(f(x))f'(x)\Delta x + g(f(x))$$

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&= g'(f(x))f'(x)
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Really, the proof of the Chain Rule is essentially a nested set of two Taylor Series approximations, one for ***f*** and the other for ***g***.

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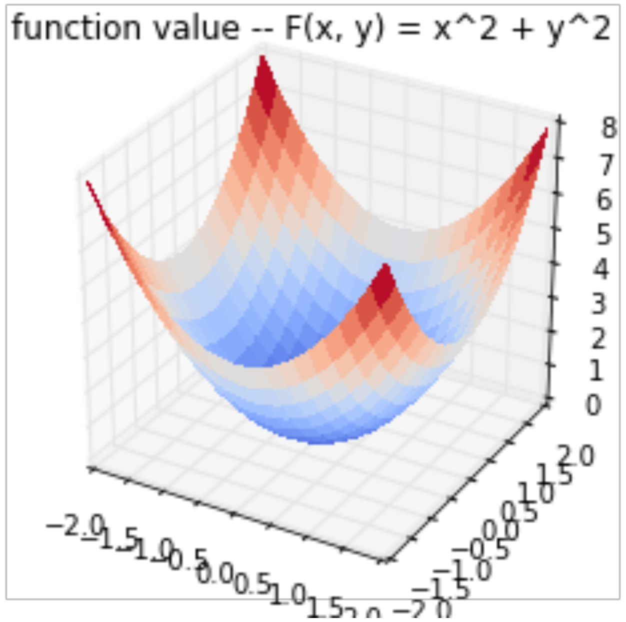
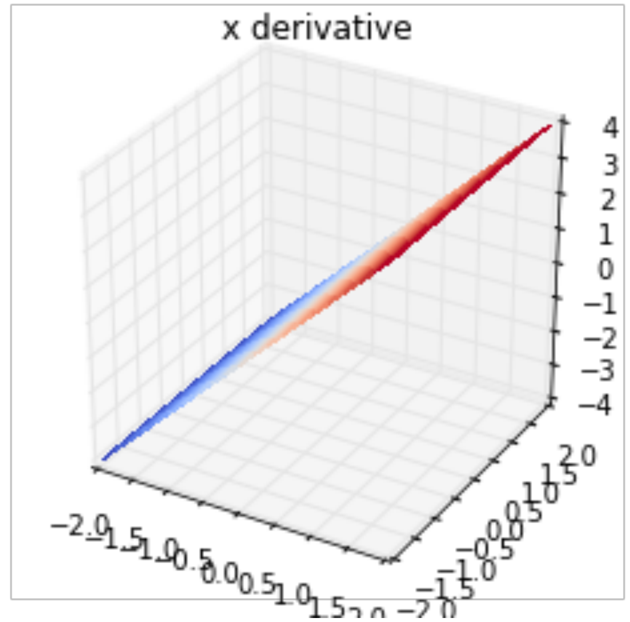
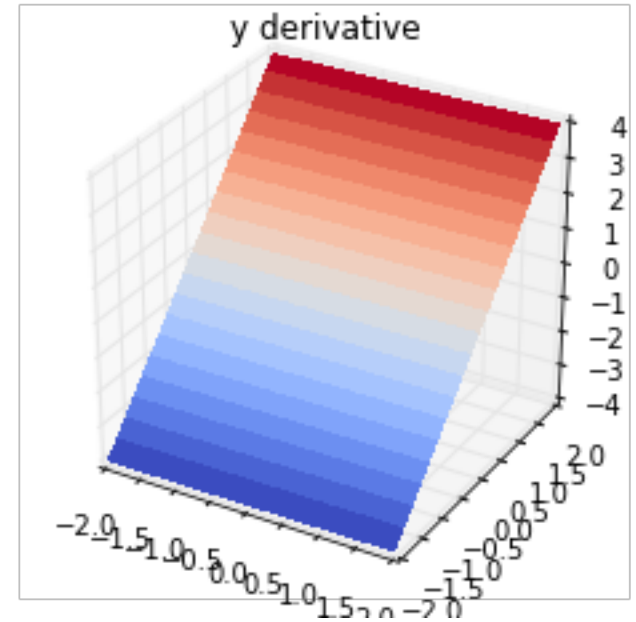
Partial derivative is w.r.t. ***x*** is just derivative of the function holding **y** constant

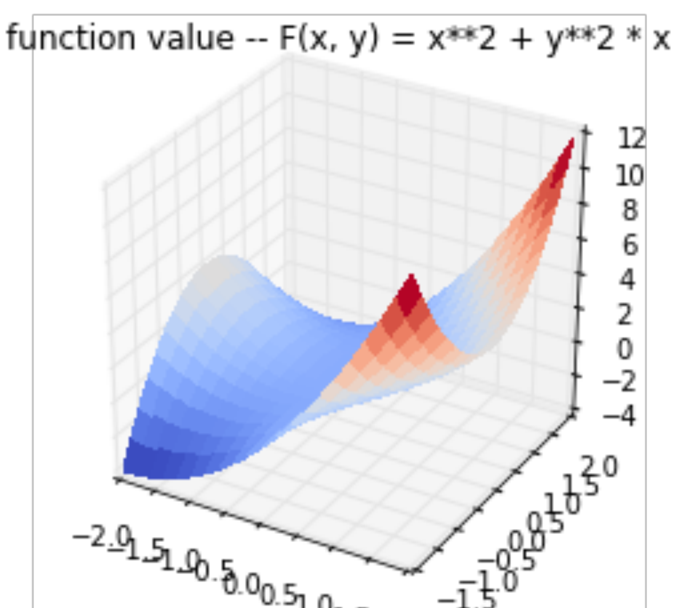
… and vice versa for partial derivative w.r.t. **y**

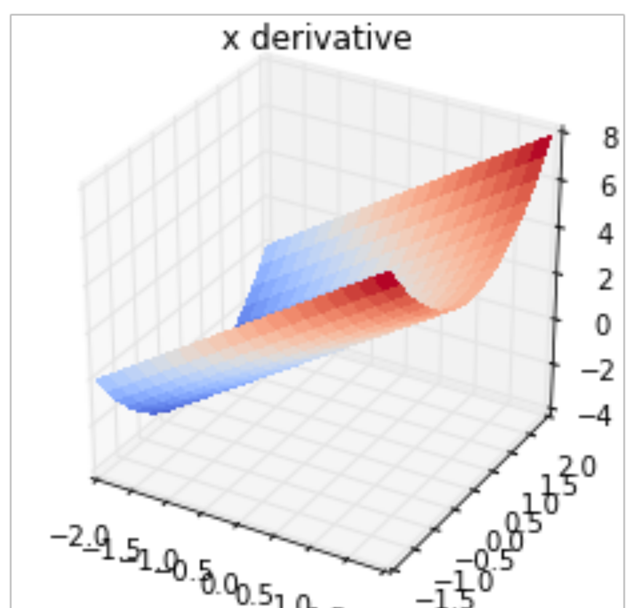
# Multi-variable Functions

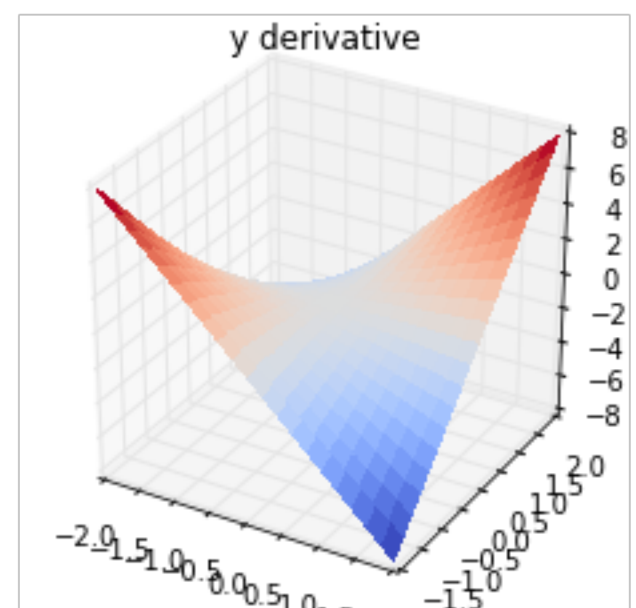


We can also calculate second partial derivatives:

$$F_{xx} = \frac{\partial^2 F(x,y)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)(x,y)$$

$$F_{xy} = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)(x,y)$$

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Let ***F(x1, x2, ..., xn)*** be a real-valued function of **n** variables.

The "gradient of ***F***" is the ***n***-vector-valued function:

$$\nabla F(\vec{x}) = \left[\frac{\partial F}{\partial x_1}(\vec{x}), \ldots, \frac{\partial F}{\partial x_n}(\vec{x})\right]$$

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the "hessian" matrix

$$H[f] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

shape = (***n***, ***n***)

# Multi-variable Functions

function itself ("0th"-derivative):

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}$$

gradient (1st derivative):

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# Multi-variable Functions

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can be generalized to k-tensor-valued function for k-th derivative ... but third derivatives and higher essentially never used...

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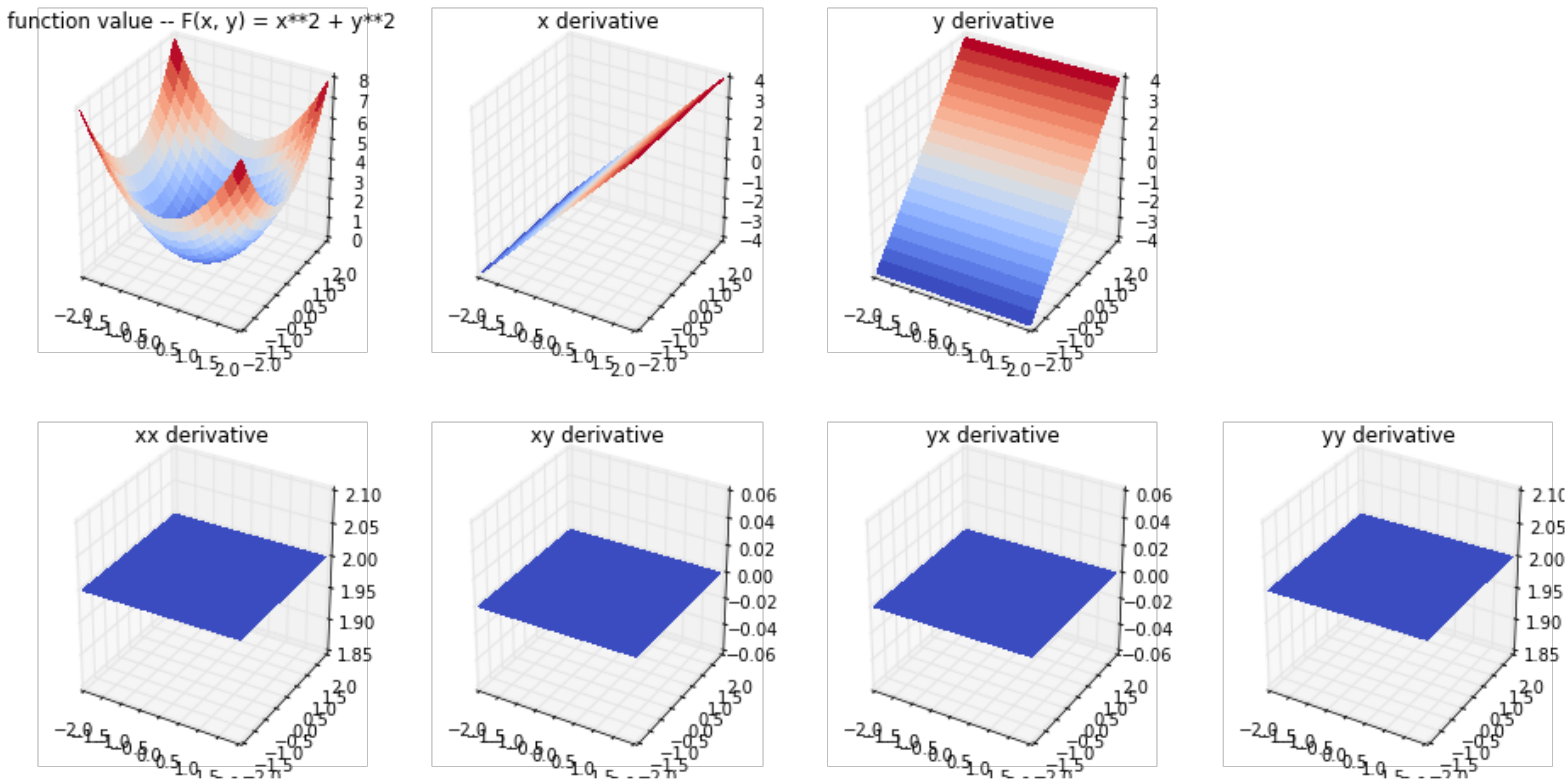

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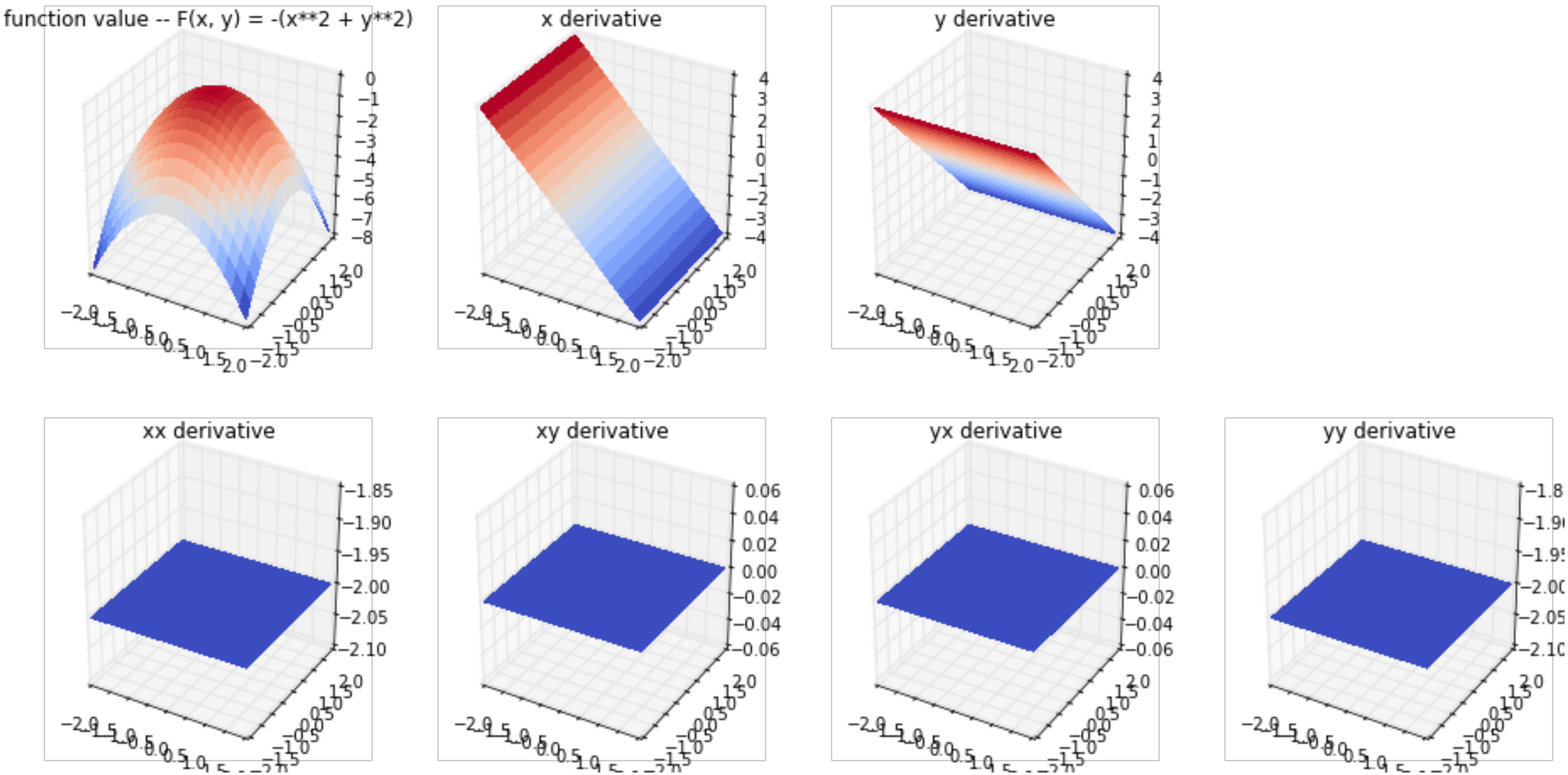

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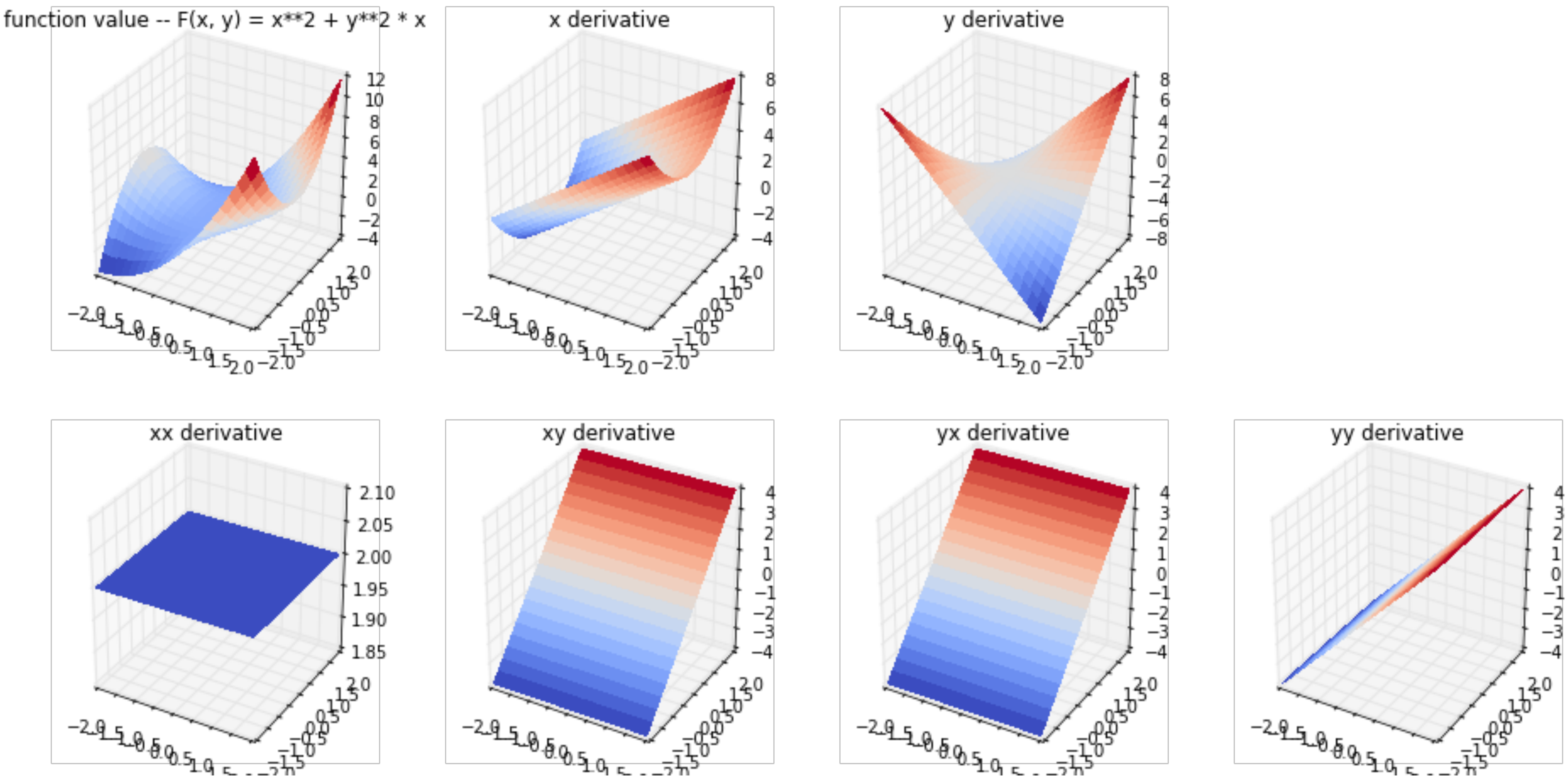

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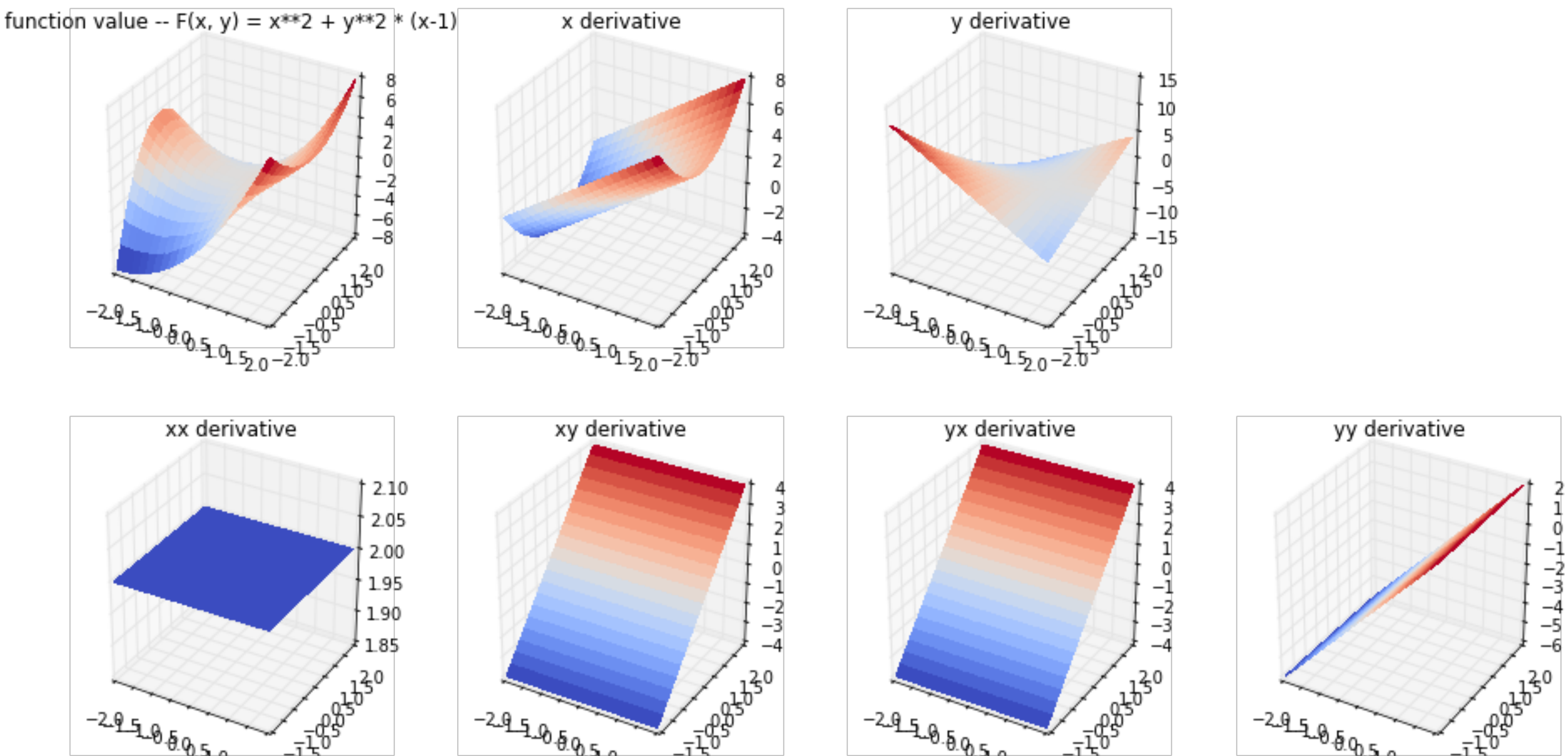


# Multi-variable Functions

In one dimension, critical points are those ***x**** such that:

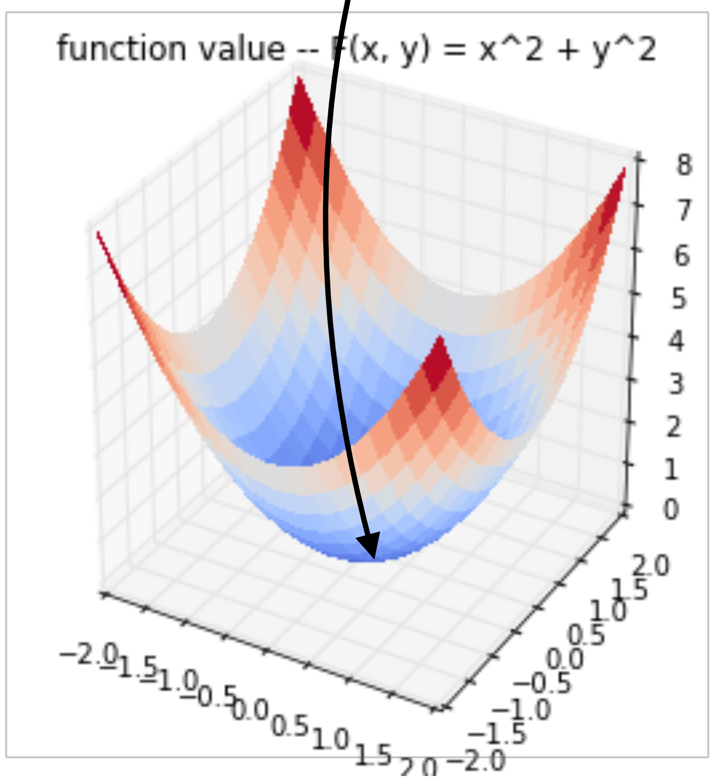
$$\left.\frac{dF}{dx}\right|_{x=x^*} = 0$$

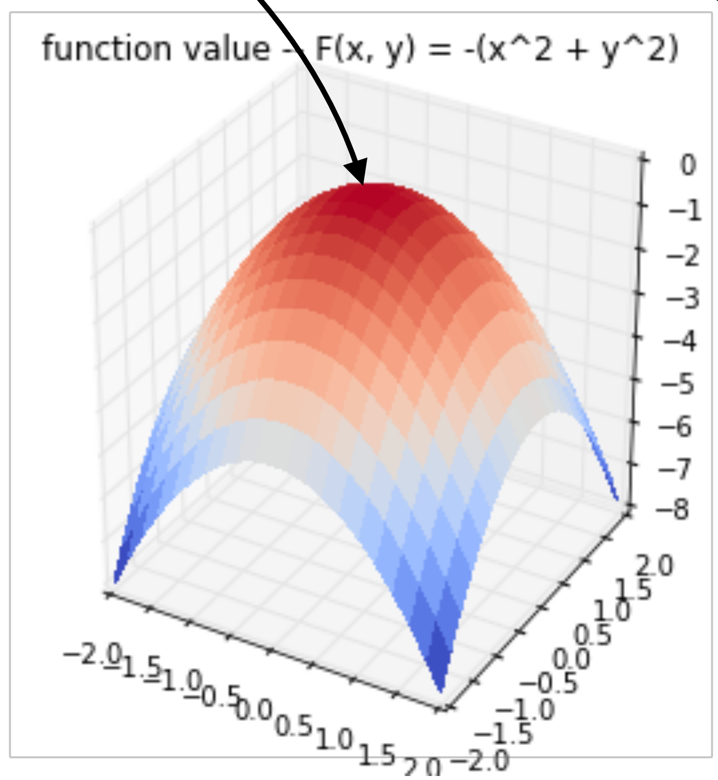
In multiple dimensions, critical points are those ***x**** such that:

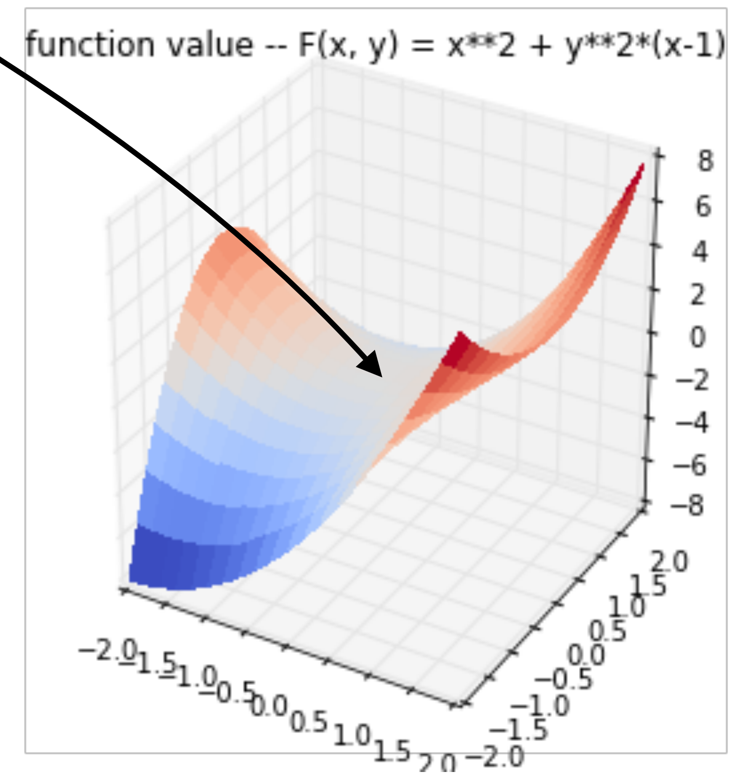
$$\nabla F(\vec{x}^*) = \left[\frac{\partial F}{\partial x_1}(\vec{x}^*), \ldots, \frac{\partial F}{\partial x_n}(\vec{x}^*)\right] = [0, \ldots, 0] = \vec{0}$$

# Multi-variable Functions

Critical point



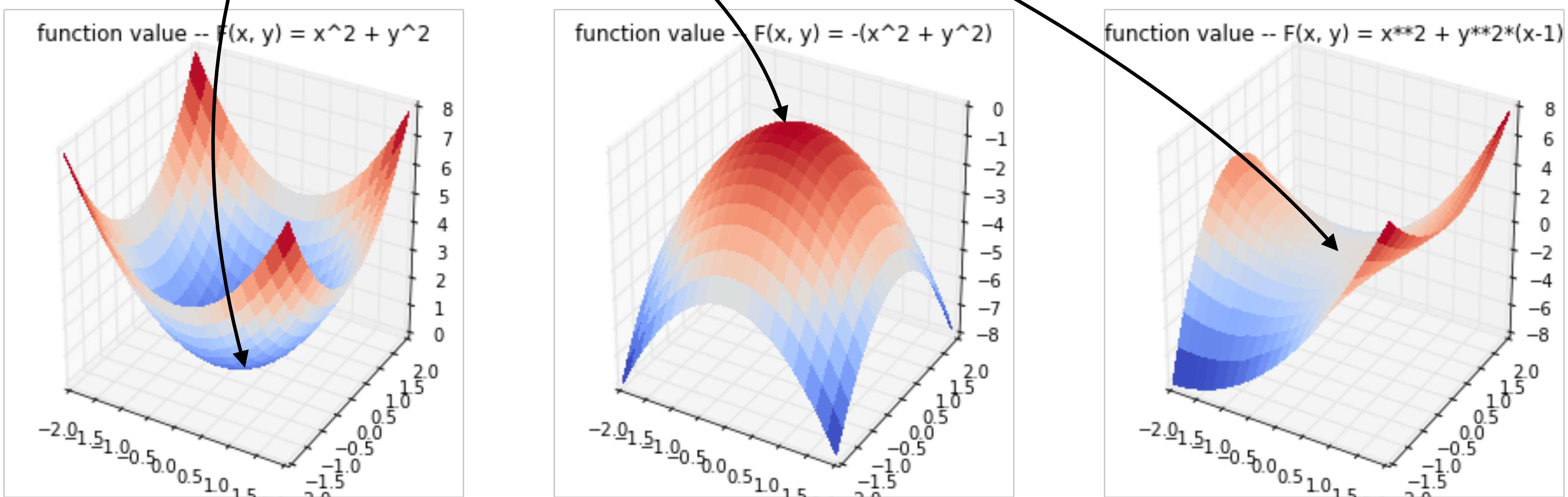



# Multi-variable Functions

Critical point

But what's the difference?




Hessian determinant positive

Hessian determinant positive

Hessian determinant negative

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What's its derivative?

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}(x_1(t + \Delta t), \ldots, x_n(t + \Delta t)) - F(x_1(t), \ldots, x_n(t))}{\Delta t}$$

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Answer:

$$\frac{df}{dt} = \frac{\partial F}{\partial x_1} \cdot \frac{dx_1(t)}{dt} + \ldots + \frac{\partial F}{\partial x_n} \cdot \frac{dx_n(t)}{dt} = \nabla F(x(t)) \cdot \frac{d\vec{x}(t)}{dt}$$

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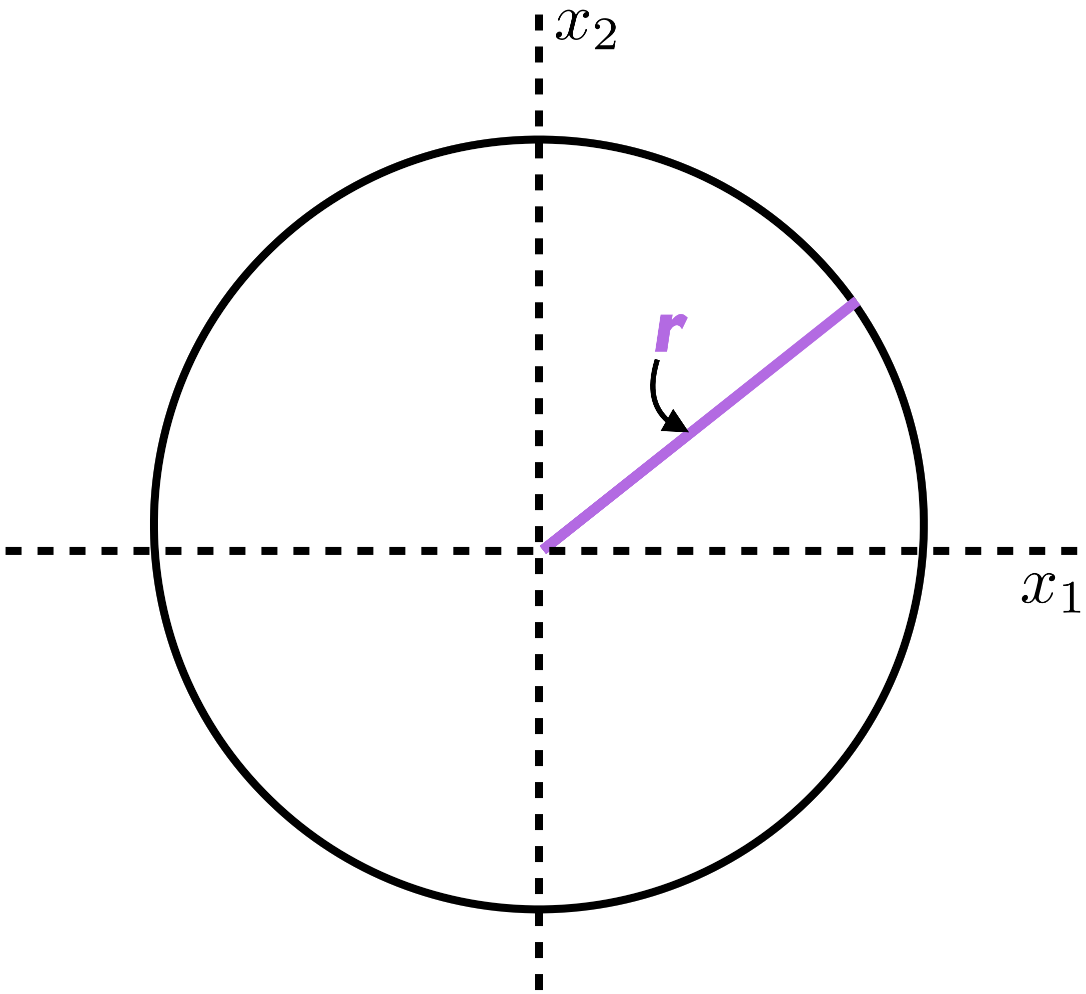

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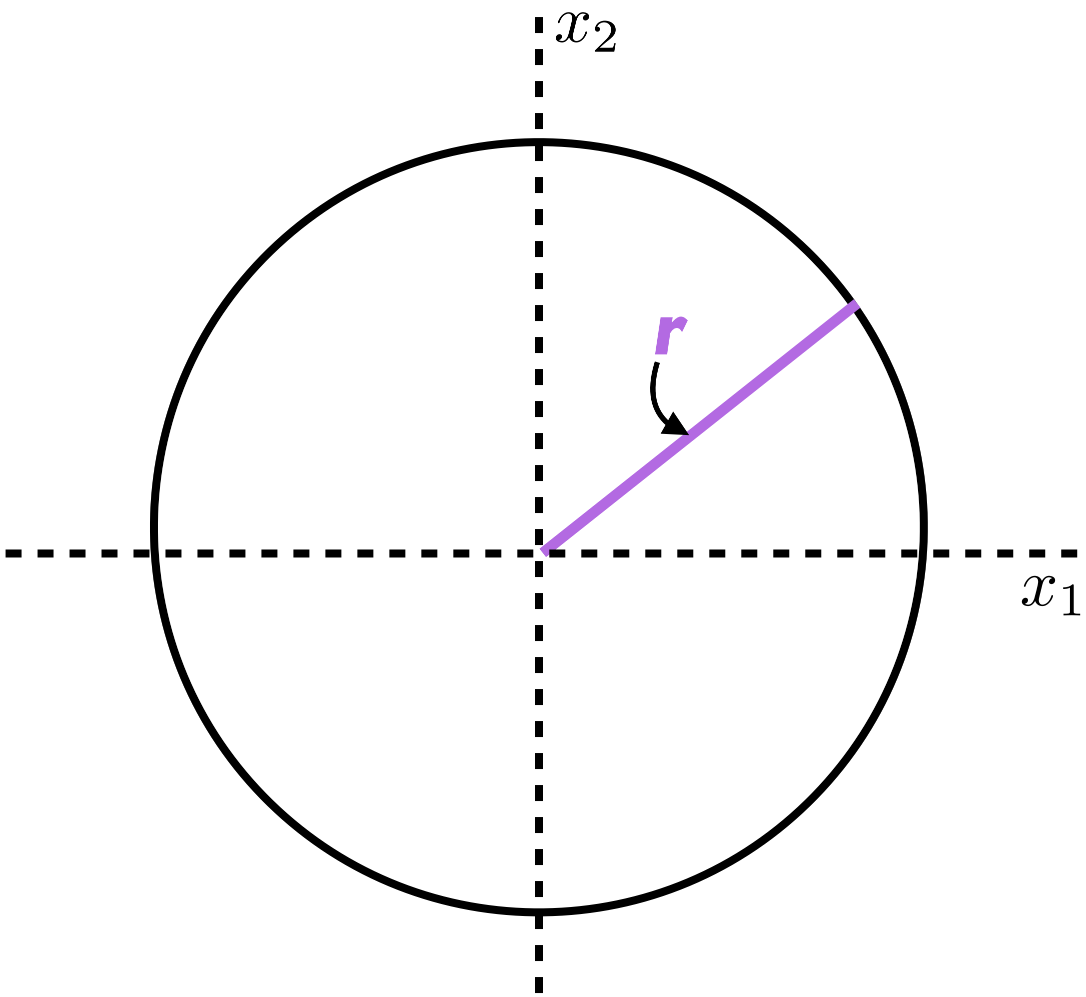


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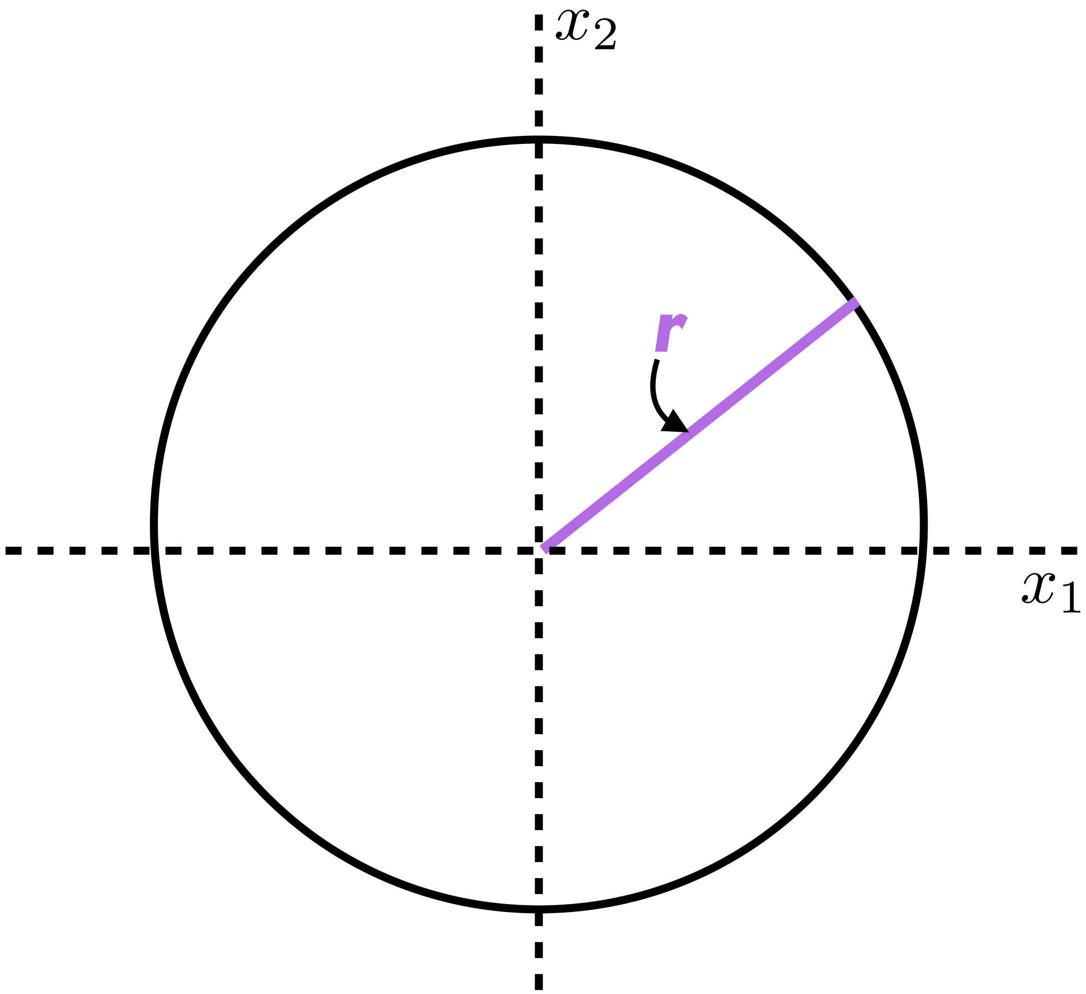


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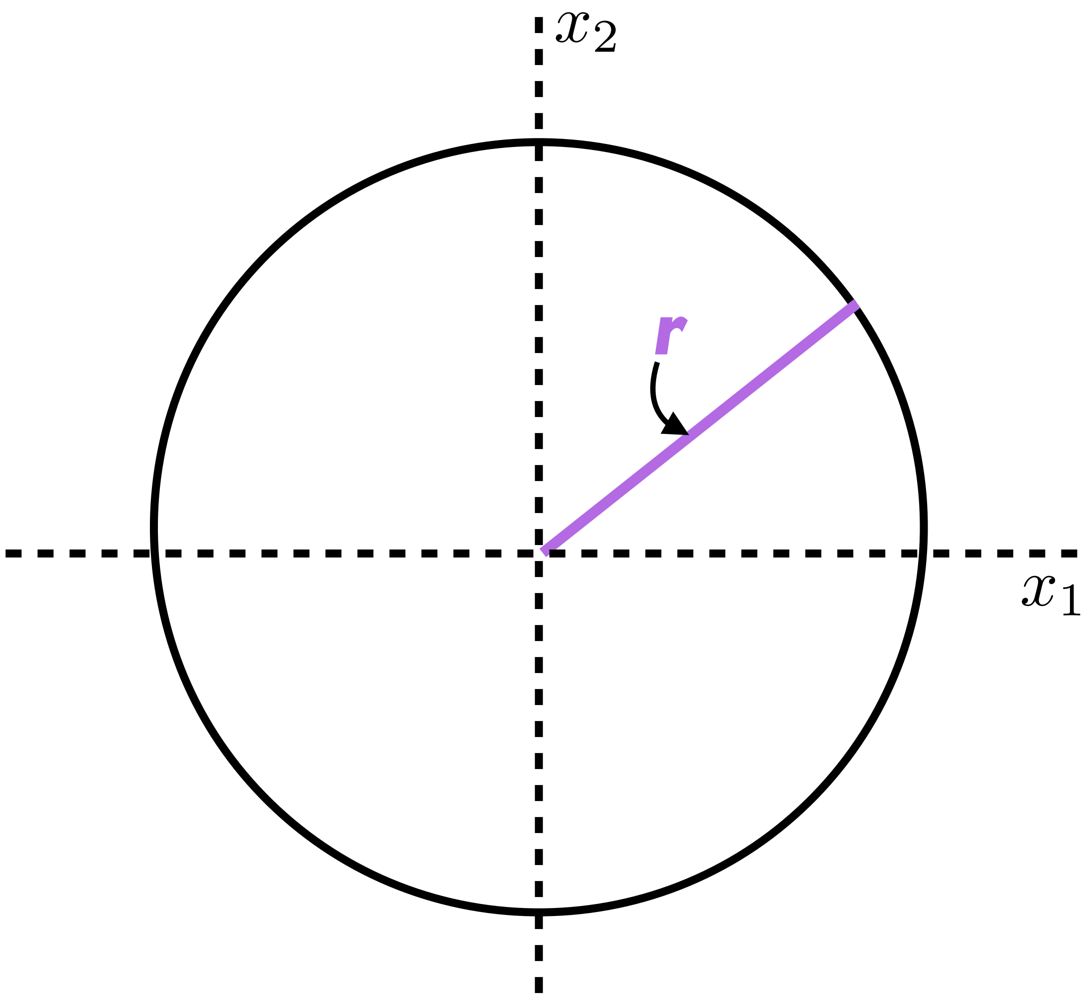


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Thus: $$\frac{df}{dt} = \frac{d}{dt}[F(x(t))] = 0$$ Not a coincidence.

# Multi-variable Functions

In general if a path ***x(t)*** travels along an isocline of ***F***, then:

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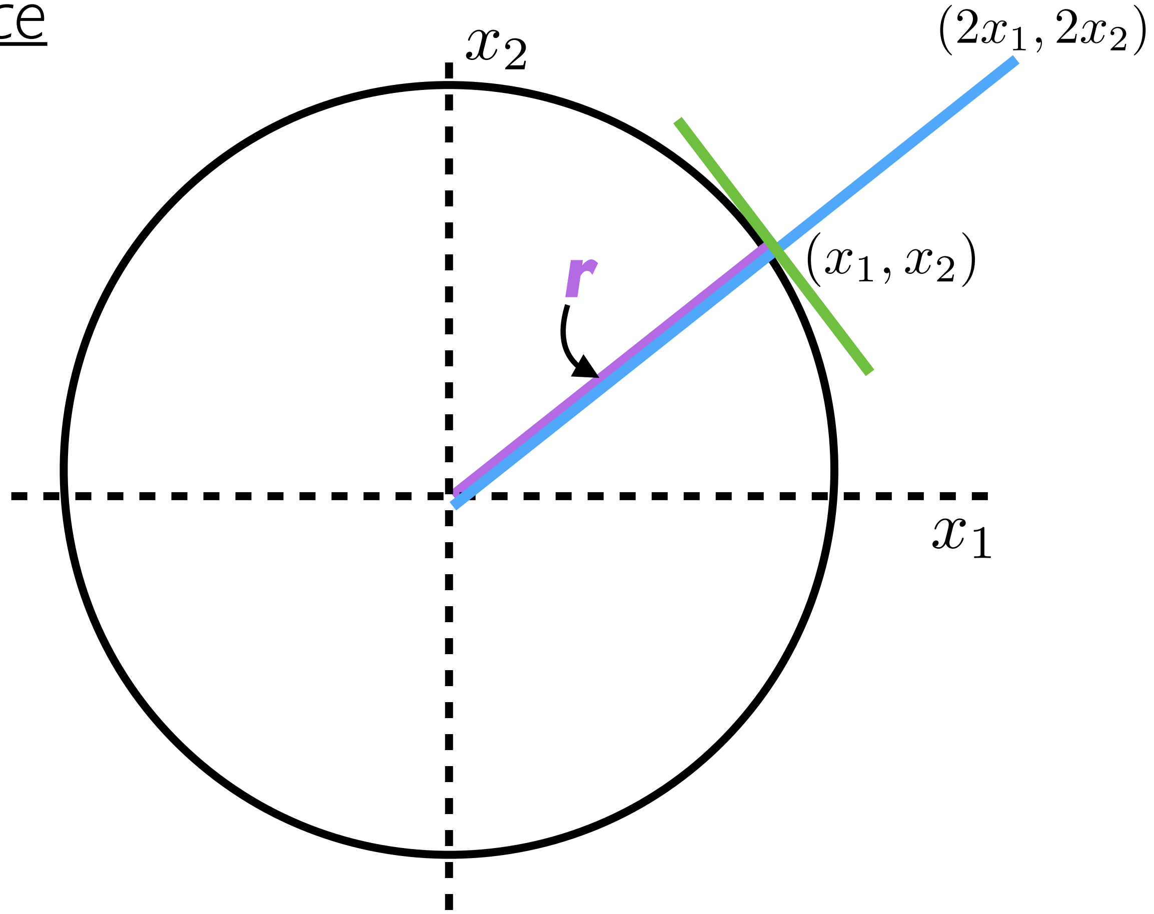
So in our example:

function: $F(x_1, x_2) = x_1^2 + x_2^2$

isocline: $F(x_1, x_2) = r^2$ *(circle of radius r)*

gradient: $\nabla_x F(x_1, x_2) = (2x_1, 2x_2)$



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E.g. suppose we want to maximize

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Subject to:

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We can't just seek critical points of ***F*** because they might not satisfy the constraint.

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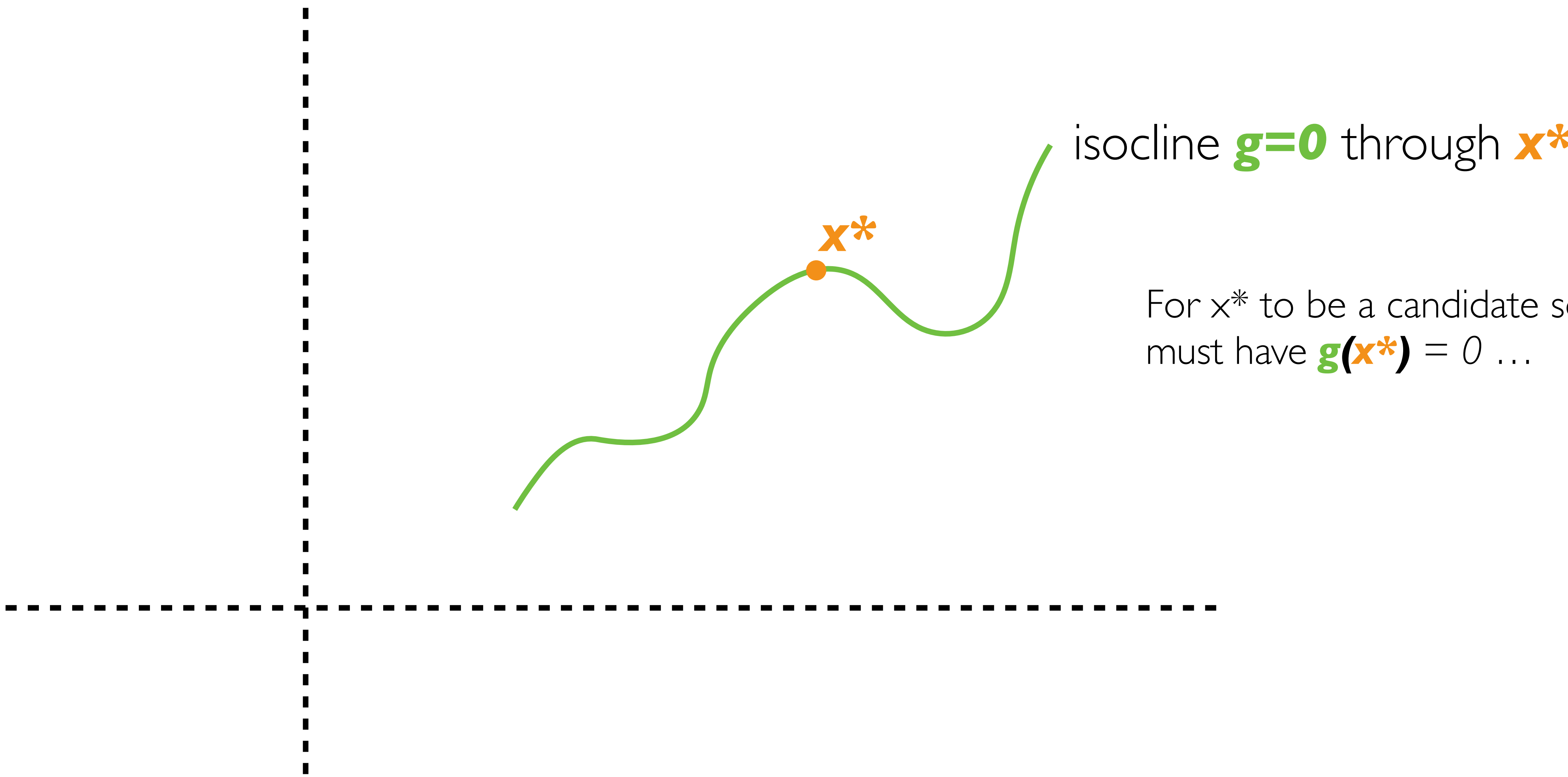


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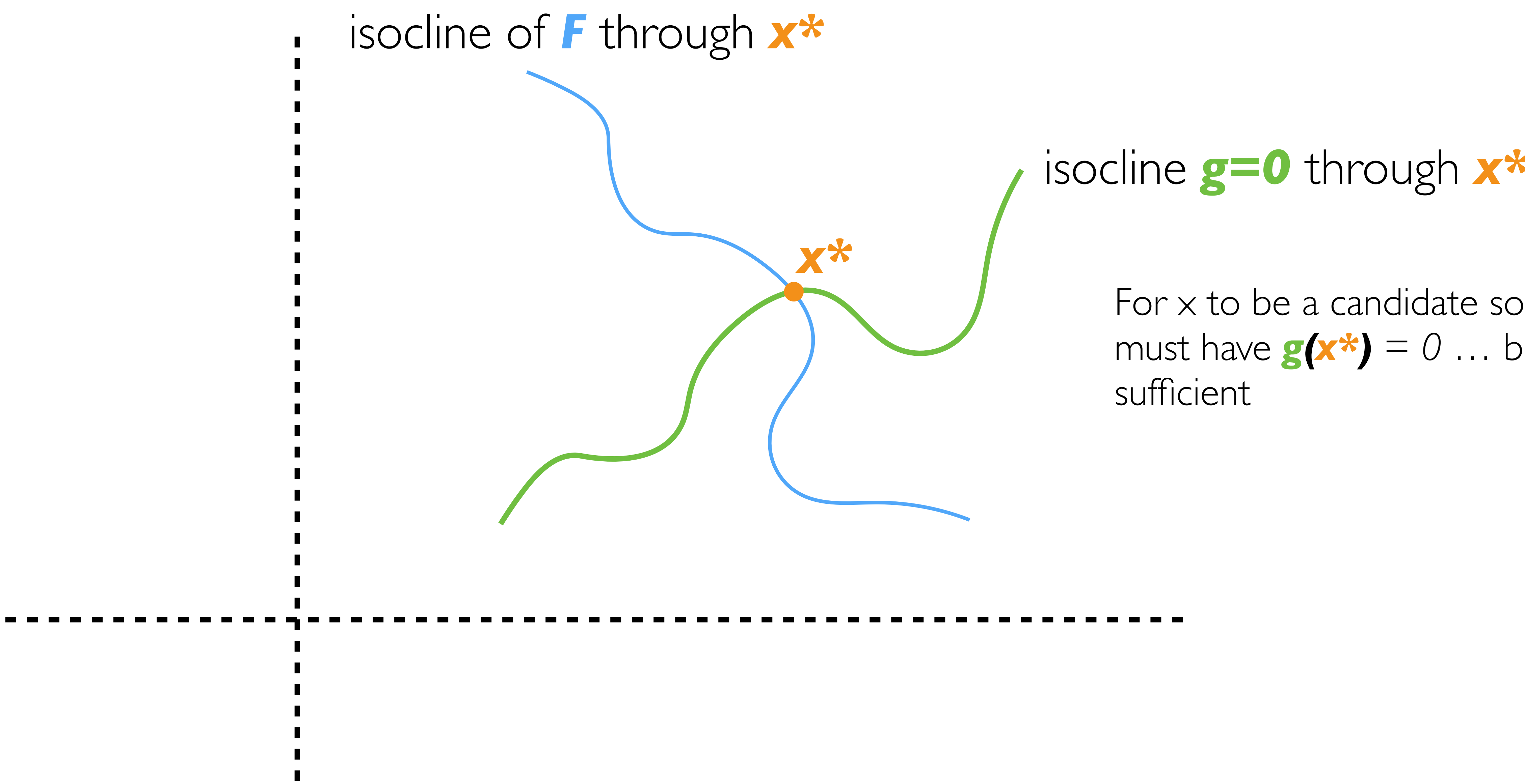


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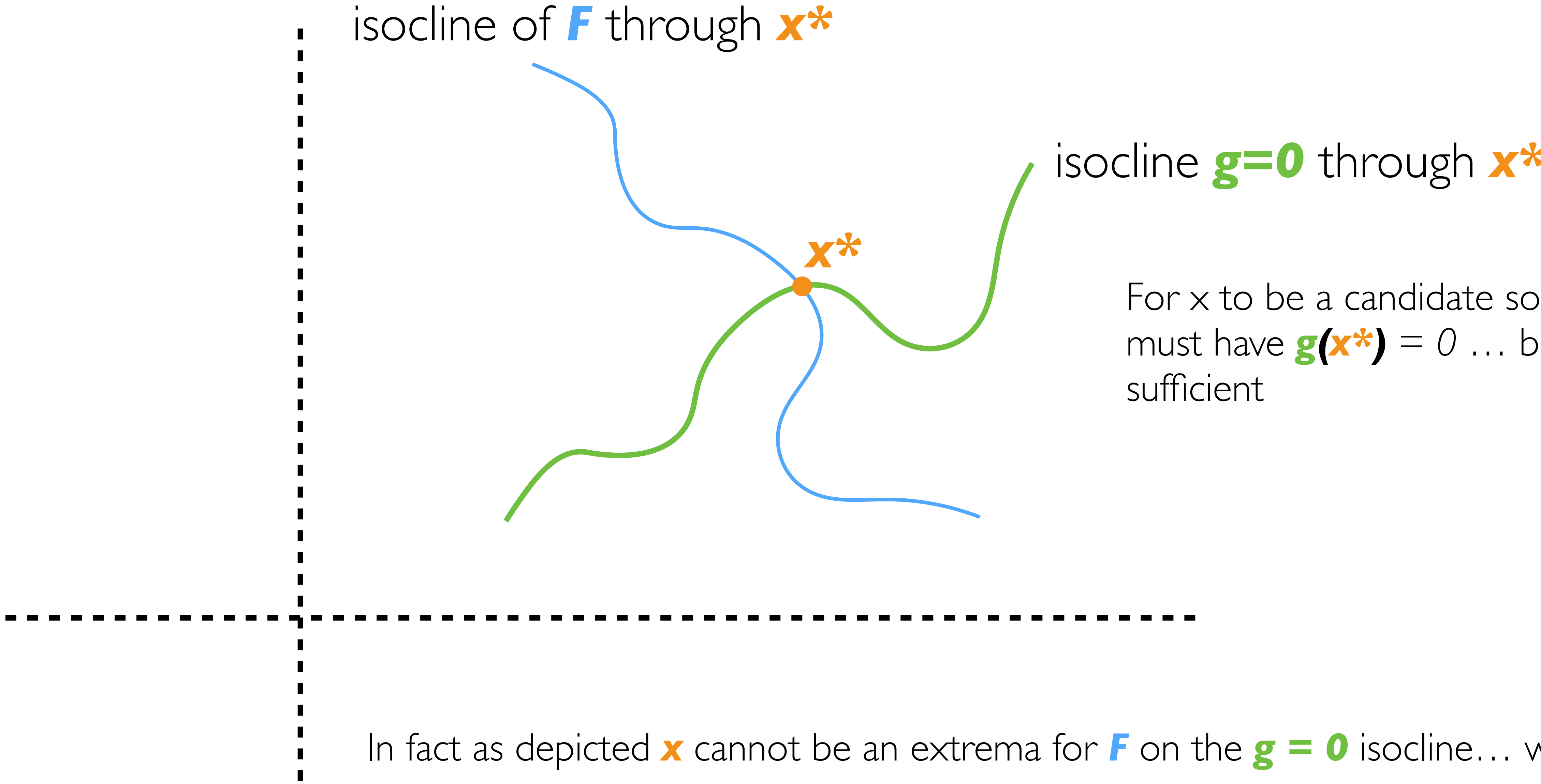


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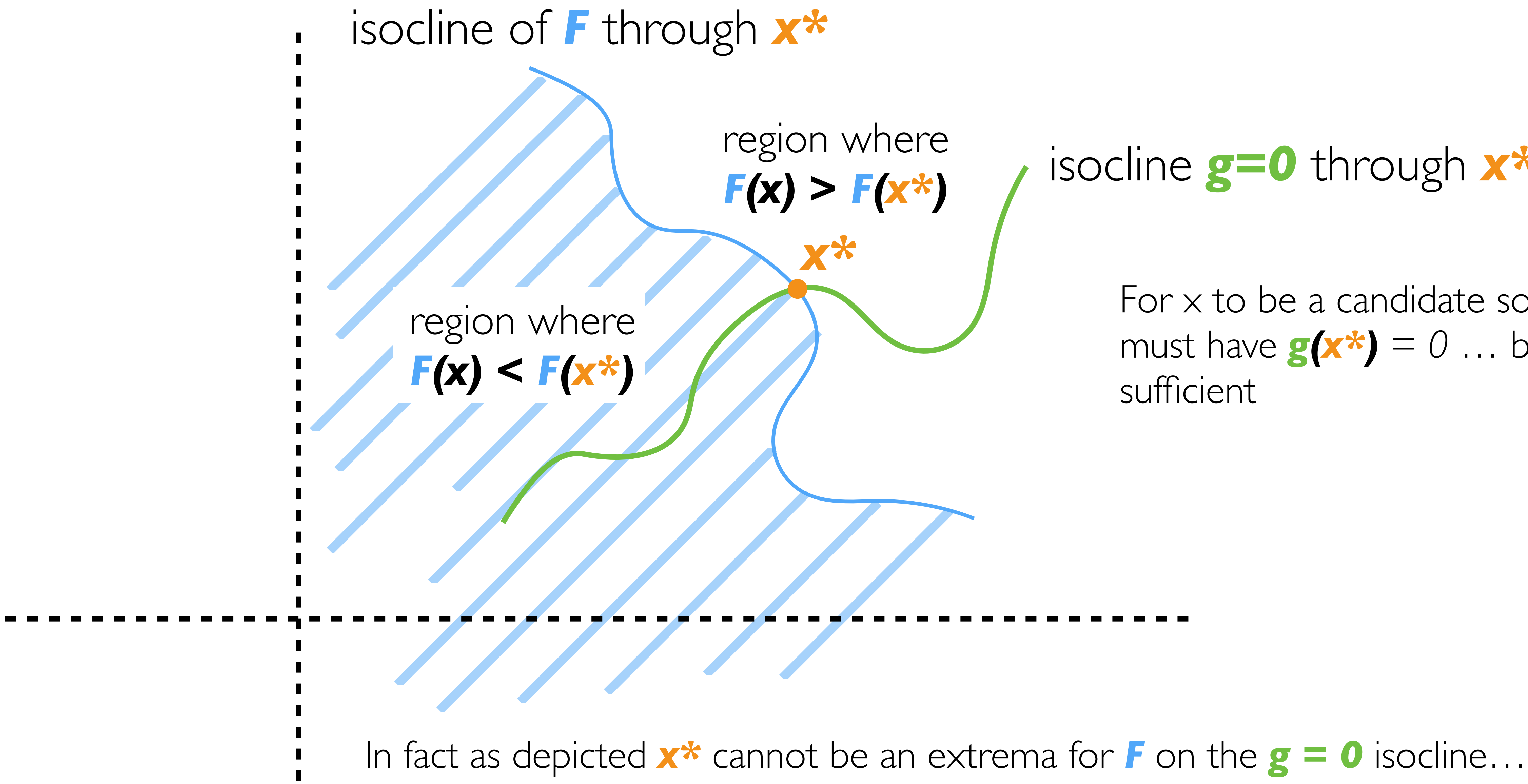


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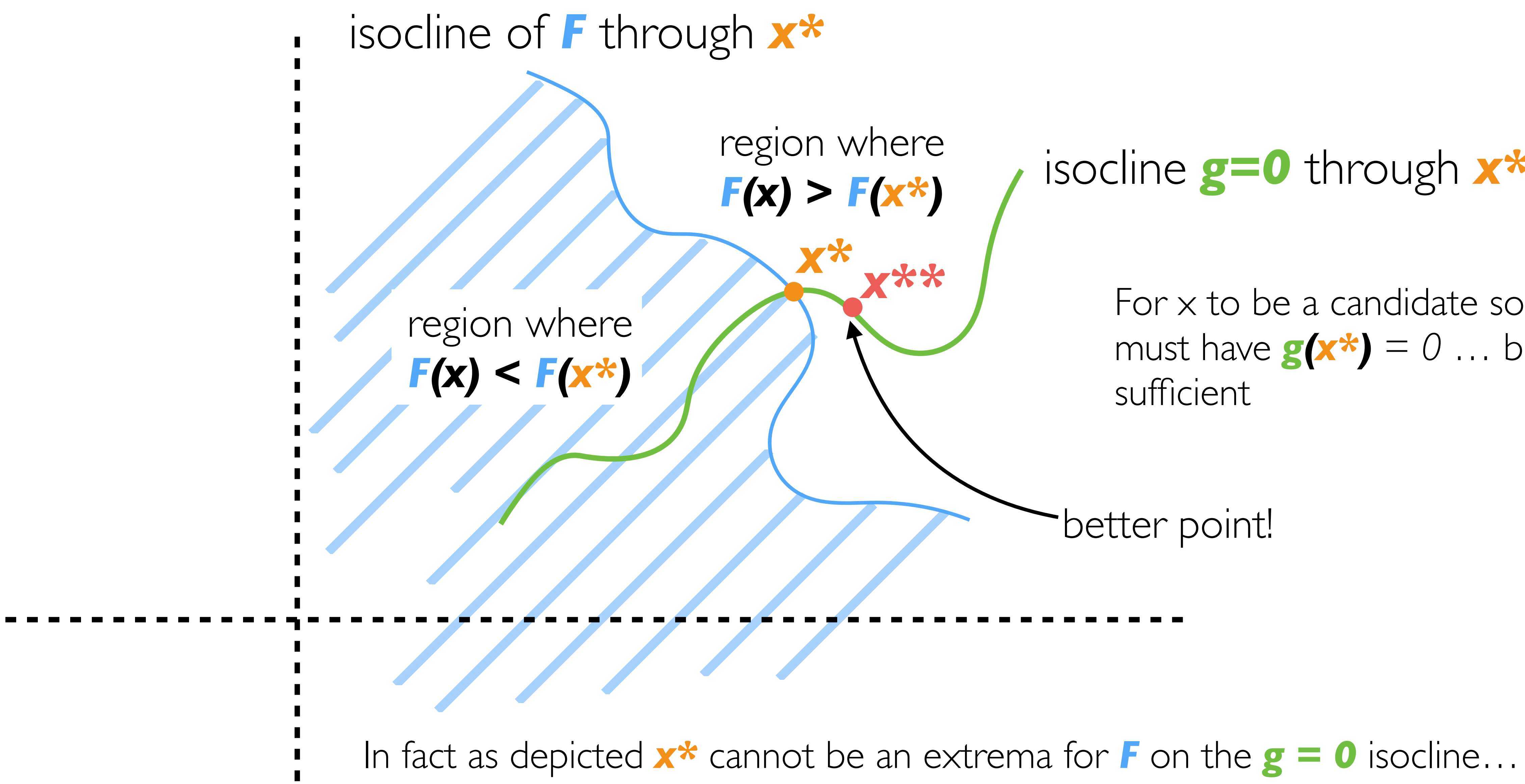


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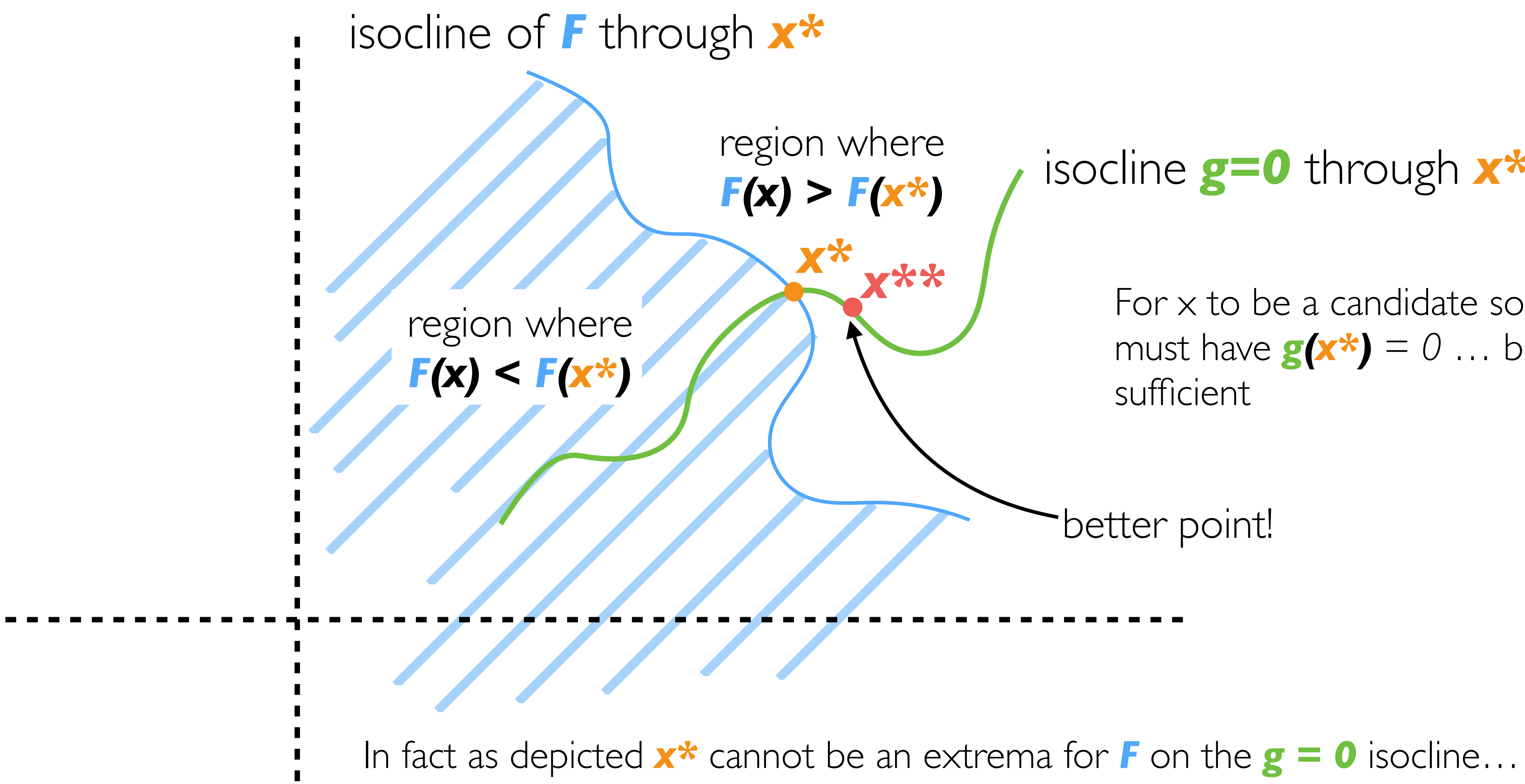
So: <u>true optima ***x**** must have isocline of ***F*** through ***x**** be parallel to ***g = 0*** isocline</u>

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# Multi-variable Functions

What if our goal is to maximize ***F*** subject to the constraint that ***g* = *0***?

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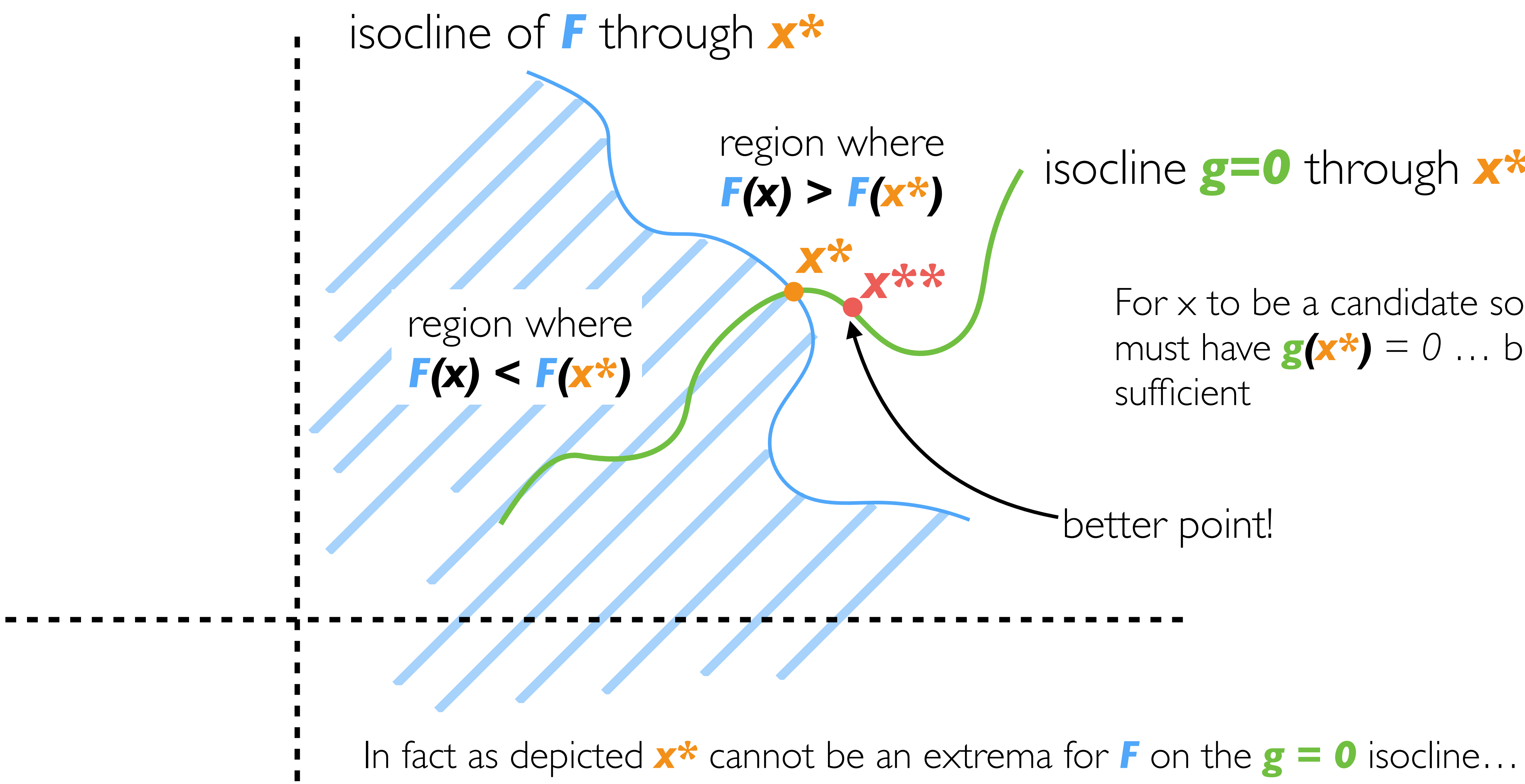


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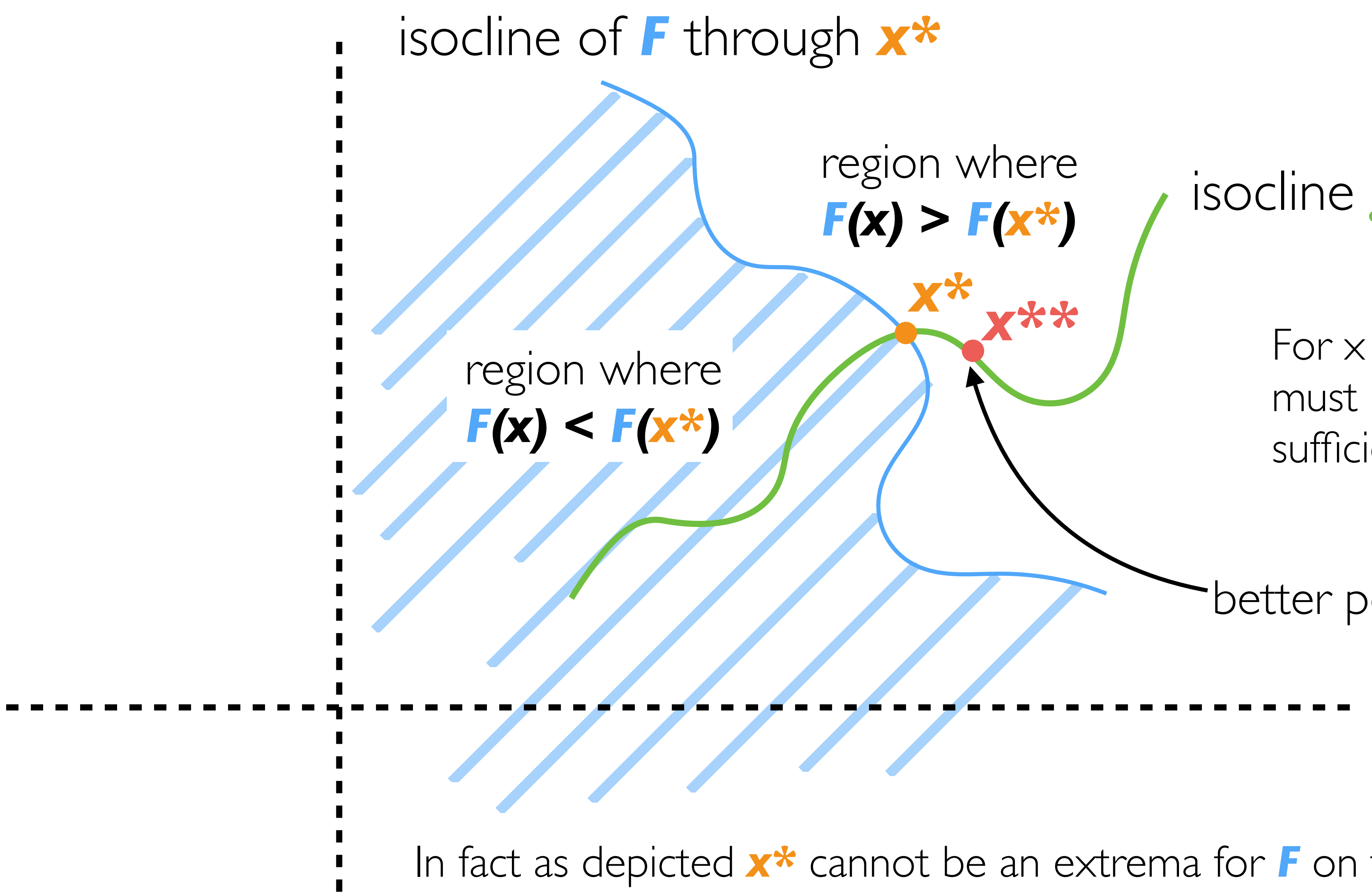


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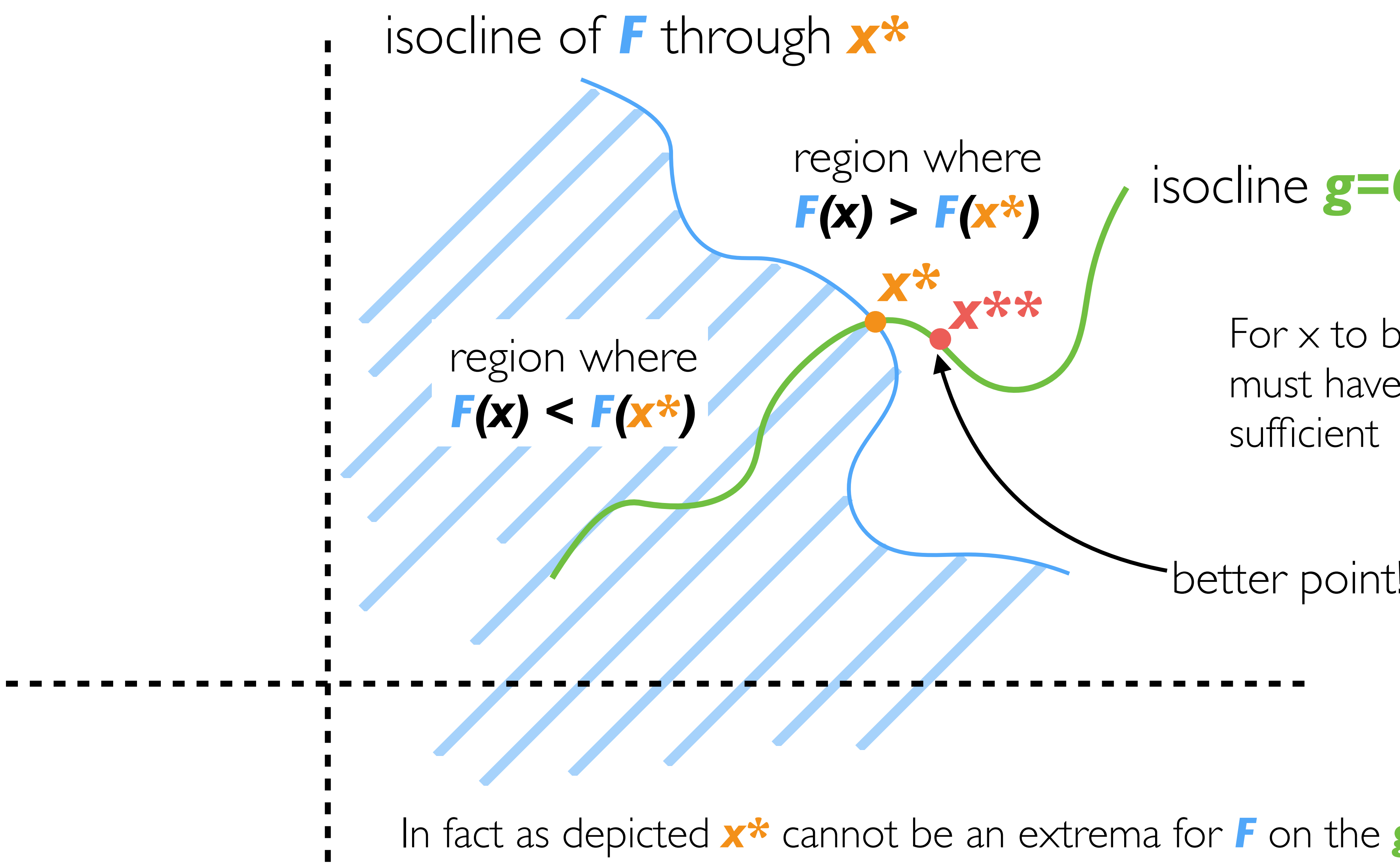


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scalar

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$$= 0 \quad \Rightarrow \quad x_1 = \frac{1}{2\lambda}, x_2 = \frac{1}{2\lambda}, \lambda = \pm\frac{1}{\sqrt{2}r}$$

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$$\Rightarrow \quad \vec{x}^* = \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right) \text{ or } \left(-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$$

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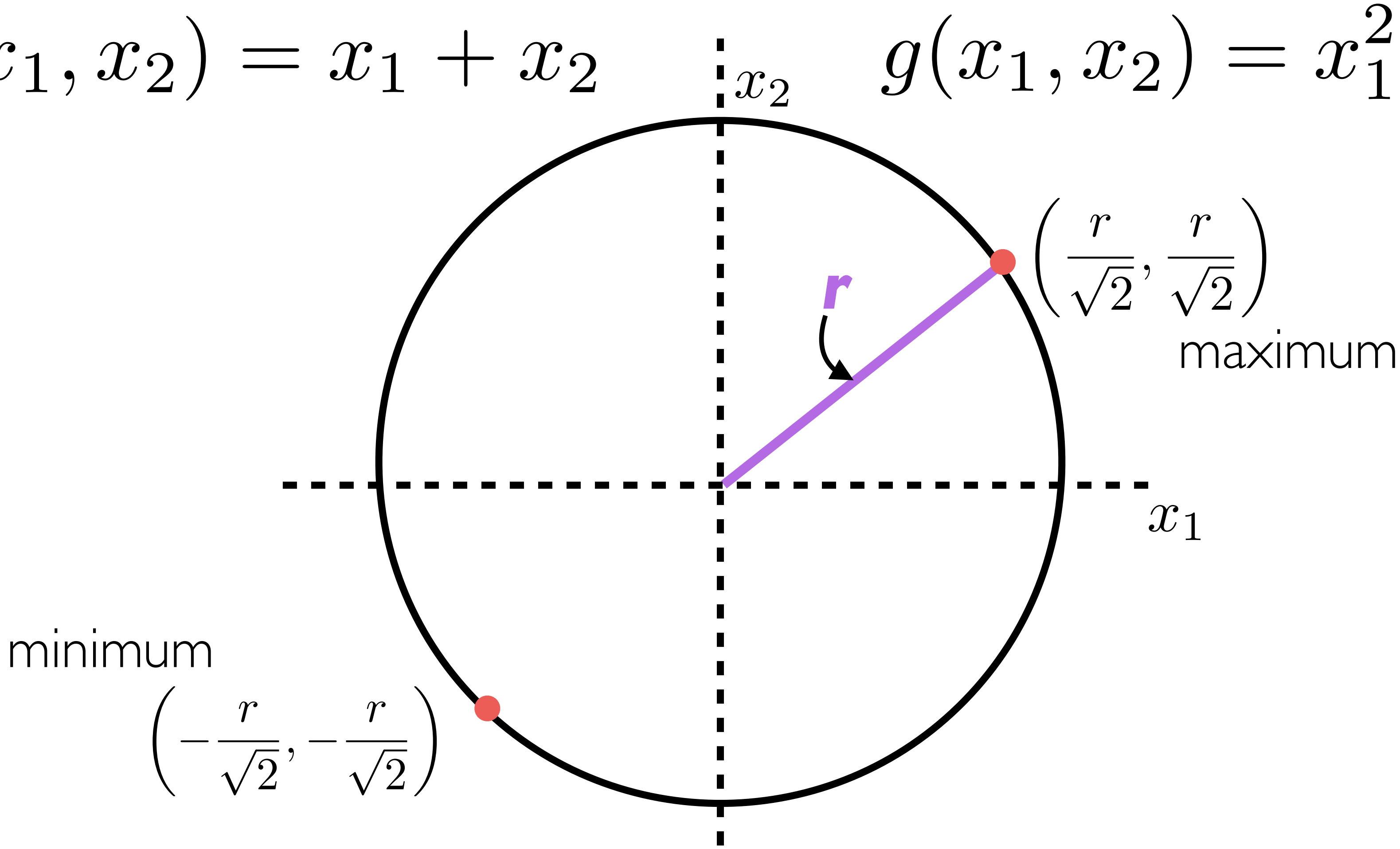

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*applying single-variable chain rule* ***m*** *times*

$$= \sum_i \frac{\partial G}{\partial y_i}(\vec{F}(\vec{x}))\frac{\partial F_i}{\partial x_1}(\vec{x})$$

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So in general:

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Or:

$$\nabla H = \nabla \vec{F}(\vec{x}) \cdot \nabla G(\vec{F}(\vec{x}))$$

***n***-vector-valued fn ***nxm-***matrix-valued fn ***m-***vector-valued fn

Where $\nabla \vec{F}$ is the matrix with ***ij***-th elt:

$$(\nabla \vec{F})_{ji} = \frac{\partial F_i}{\partial x_j}$$

# Multi-variable Functions

Taylor series for function ***f(x1, ... xn)*** of ***n*** inputs:

pointwise mult.

$$f(\vec{x}_0, \Delta\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) * \Delta\vec{x} + \frac{1}{2}(H[F] \cdot \vec{x}_0) * \Delta x^2 + \ldots$$

vector delta

matrix-on-vector multiplication

Hessian matrix

$$H[f] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

shape = (***n***, ***n***)