Challenge: Diffusion

- 1D Gaussian Diffusion: \( \Delta l = \sqrt{2D \Delta t} \)

- Imagine a sequence with 2 gradients of area GT, with a 180 refocusing pulse between.

- What is the expected value of the spin echo signal as a function of D, \( \Delta t \), GT, ignoring \( T_2 \)?
Challenge: Diffusion (Solution)

- Phase vs \( x \) is \( \phi = \gamma GT \, x \)

- Expected value is expected value of \( \cos(\phi) \)

\[
\int \cos(\gamma GT \, x) \frac{1}{\sqrt{4\pi D \Delta t}} e^{-\frac{x^2}{4D\Delta t}} \, dx = \frac{\sqrt{4\pi D \Delta t}}{\sqrt{4\pi D \Delta t}} e^{-\gamma^2 GT^2 \, D \Delta t} = e^{-bD}
\]

\[
b = (\gamma GT)^2 \Delta t
\]

\[
\sigma = \Delta l = \sqrt{2D\Delta t}
\]

\[
\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}
\]
Extended Phase Graphs (EPG)

- Purpose / Definition
- Propagation
  - Gradients, Relaxation, RF
  - Diffusion
- Examples
**EPG Motivating Example: RF with Crushers**

- Crushers are used to suppress spins that do not experience a 180° rotation.
- Dephasing/Rephasing work if 180° is perfect.
EPG Motivating Example: RF with Crushers

$G_z$  $120^\circ$ RF  $G_z$

$M_y$  $M_x$  $M_y$  $M_x$  $M_y$

180 Degree Refocusing Angle

Spin Echo vs Refoc. Angle
EPG Motivating Example: RF with Crushers

- Brute-force simulation works, but little intuition
- Quantized gradients produce “dephased cycles”
- Decompose elliptical distributions to +/- circular
Extended Phase Graphs: Purpose

If we assume...

Gradient areas are quantized into units that give a phase twist of one cycle (2π) across a voxel

... then we can easily represent large groups of spins with a simple Fourier basis, and accurately calculate MR signals


Hennig J. et al. Calculation of flip angles or echo trains with predefined amplitudes with the extended phase graph (EPG)-algorithm: principles and applications to hyperecho and TRAPS sequences. MRM 2004; 51:68-80

Weigel M. Extended Phase Graphs: Dephasing, RF Pulses, and Echoes - Pure and Simple. JMRI 2014;
Basic Magnetization Vector Definitions

- Common representations for magnetization:

\[ M_{xy} = M_x + iM_y \quad M_{xy}^* = M_x - iM_y \]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
= 
\begin{bmatrix}
0.5 & 0.5 & 0 \\
-0.5i & 0.5i & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
M_{xy} \\
M_{xy}^* \\
M_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{xy} \\
M_{xy}^* \\
M_z
\end{bmatrix}
= 
\begin{bmatrix}
1 & i & 0 \\
1 & -i & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]
The EPG Basis

- Consider spins in a voxel:
  - $z$ is the location (0 to 1) across the voxel
  - $M_{xy}(z)$ and $M_{xy}^*(z)$ are the transverse magnetization
  - $M_z(z)$ is the longitudinal magnetization
- Represent the (huge) number of spins using a compact basis set of $F_n$ and $Z_n$ coefficients

**Motivation:** Propagation of the basis functions through RF, gradients, relaxation is fairly simple
• $F_n$ and $Z_n$ are basis coefficients

• Transverse basis are simple phase twists (*sign of n indicates direction*)

• Longitudinal basis are sinusoids
EPG Basis: Mathematically

- Transverse basis functions \((F_n)\) are just phase twists:
  \[
  M_{xy}(z) = \sum_{n=-\infty}^{\infty} F_n^+ e^{2\pi i n z} + \sum_{n=1}^{\infty} \left[ F_n^+ e^{2\pi i n z} + (F_n^-)^* e^{-2\pi i n z} \right]
  \]

- Longitudinal basis functions \((Z_n)\) are sinusoids:
  \[
  M_z(z) = \text{Real} \left\{ Z_0 + 2 \sum_{n=1}^{N} Z_n e^{2\pi i n z} \right\}
  \]

\(F_n\) and \(Z_n\) are the coefficients, but we sometimes use them to refer to the basis functions ("twists") they multiply.

Although there are other basis definitions, this is consistent with that of Weigel et al. J Magn Reson 2010; 205:276-285.
Magnetization to EPG Basis

• $F^+$ states:

$$F^+_n = \int_0^1 M_{xy}(z)e^{-2\pi inz} \, dz$$

• $F^-$ states:

$$F^-_n = F^*_{-n} = \int_0^1 M^*_x(z)e^{-2\pi inz} \, dz$$

• $Z$ states:

$$Z_n = \int_0^1 M_z(z)e^{-2\pi inz} \, dz$$

Note redundancy $F^-_n = (F^+_n)^*$

(Can use $F^+_n$ and $F^-_n$ for $n>0$, or just $F^+_{-n}$ (all $n$))
Review Question

• What are the $F^+$ and $F^-$ states that represent this magnetization (entirely along $M_y$)?

Recall:

$$M_{xy}(z) = \frac{1}{2} \sum_{n=1}^{\infty} F^n e^{2\pi i n z}$$
Basis Functions in MR Sequences

**Key Point:** Basis is easily propagated in MR sequences:

- **Gradient (one cycle over voxel):**
  - Increase/decrease $F^+_n$ or $F^-_n$ state number (n), e.g. $F^+_{n+1}=F^+_n$

- **RF Pulse**
  - Mixes coefficients between $Z_n$, $F^+_n$, and $F^-_n$ [ or $(F^{+n})^*$ ]

- **Relaxation**
  - $T_2$ decay attenuates $F_n$ coefficients
  - $T_1$ recovery attenuates $Z_n$ coefficients, and enhances $Z_0$

- **Diffusion**
  - Increasing attenuation with n (described later)
**EPG Propagation: Gradient**

Gradients induce one cycle \((2\pi)\) of phase across a voxel

\[ F_{n+1}^+ = F_n^+ \]

- Magnetization in \( F_n^+ \) moves to \( F_{n+1}^+ \)
  - Dephasing for \( n \geq 0 \)
  - Rephasing for \( n < 0 \) (or \( F^- \))
- Generally can apply \( p \) cycles (increase \( n \) by \( p \))
- \( Z \) states are unaffected

Assume the “gradient” such as a spoiler or crusher induces one twist cycle per voxel
EPG Relaxation over Period T

• Transverse states:
  \[ F_n' = F_n e^{-T/T2} \]

• \( Z_n \) states attenuated:
  \[ Z_n' = Z_n e^{-T/T1} \quad (n>0) \]

• \( Z_0 \) state also experiences recovery:
  \[ Z_0' = M_0 (1 - e^{-T/T1}) + Z_0 e^{-T/T1} \]
**EPG RF: Transverse Effects**

- First consider transverse spins after one cycle of dephasing:

- After a $60^\circ$ tip, the transverse distribution is elliptical, but can be decomposed into opposite circular twists ($F^+_1$ and $F^-_1$)

\[ F^+_1 M_x = - F^-_1 M_x \]

Longitudinal (Zn) states are also affected – described shortly
EPG RF Rotations

- An RF pulse cannot change the number of cycles.
- “Mixes” $F^+_n$, $F^-_n$, and $Z_n$ (details next…)

![Diagram of EPG RF Rotations](image)
• Dephased magnetization generally has an elliptical distribution, even after additional nutations (flips)

• Can decompose into a sum of opposite circular twists (previous slide) and cosines along $M_z$

• Can derive (trigonometrically) for flip angle $\alpha$ about a transverse axis with angle $\phi$ from $M_x$:

\[
\begin{bmatrix}
F^+_{n} \\
F^-_{n} \\
Z_n
\end{bmatrix}' =
\begin{bmatrix}
\cos^2(\alpha/2) & e^{2i\phi}\sin^2(\alpha/2) & -ie^{i\phi}\sin\alpha \\
e^{-2i\phi}\sin^2(\alpha/2) & \cos^2(\alpha/2) & ie^{-i\phi}\sin\alpha \\
-i/2e^{-i\phi}\sin\alpha & i/2e^{i\phi}\sin\alpha & \cos\alpha
\end{bmatrix}
\begin{bmatrix}
F^+_{n} \\
F^-_{n} \\
Z_n
\end{bmatrix}
\]

Same $R_\phi$ RF rotation as before, in $[M_{xy},M_{xy}^*,M_z]^T$ frame
Example: $180^\circ$ Refocusing Pulse

Magnetization Starting as $F^+_1 = 1$
Phase Graph “States” (Flow Chart)
Examples:

- For a $90_x$ rotation:
  - (Right-handed!)

- For a $90_y$ rotation:
  - (Right-handed!)

\[
\begin{bmatrix}
F^+_n \\
F^-_n \\
Z_n
\end{bmatrix}' = \begin{bmatrix}
\cos^2(\alpha/2) & e^{2i\phi} \sin^2(\alpha/2) & -ie^{i\phi} \sin \alpha \\
e^{-2i\phi} \sin^2(\alpha/2) & \cos^2(\alpha/2) & ie^{-i\phi} \sin \alpha \\
-i/2e^{-i\phi} \sin \alpha & i/2e^{i\phi} \sin \alpha & \cos \alpha
\end{bmatrix}\begin{bmatrix}
F^+_n \\
F^-_n \\
Z_n
\end{bmatrix}
\]
Example 1: Ideal Spin Echo Train

- 90° excitation transfers $Z_0$ to $F_0$
- Crusher gradients cause twist cycle
- Relaxation between RF pulses (Here $e^{-\frac{TE}{T2}}=0.81$)
- $180^\circ_x$ pulse rotation:
  - Swap $F_n$ and $F_{-n}$
  - Invert $Z_n$
Example 1: Ideal Spin Echo Train

\[ \begin{align*}
\text{RF} & \quad 90^\circ & \quad 180_x^\circ & \quad 180_x^\circ \\
G_z & \quad Z_0 & \quad F^+_0 & \quad F^+_1 & \quad F^-_1 & \quad F^+_0 & \quad F^+_1 & \quad F^-_1 & \quad F^+_0
\end{align*} \]

\[ \begin{align*}
\text{Signal} & = 0.81 \\
\text{Signal} & = 0.66
\end{align*} \]

\[ Q = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0.90 \\
0 & 0 \\
0.05 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0.81 & 0 \\
0.81 & 0 \\
0 & 0 \\
-0.05 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0.73 \\
0 & 0.05 \\
0 & 0 \\
0.66 & 0
\end{bmatrix} \]
Example 1: Summarized

\[ Q = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 0 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
0 & 0.90 \\
0 & 0 \\
0.05 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0 \\
0 & 0.90 \\
-.05 & 0 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
0.81 & 0 \\
0.81 & 0 \\
0 & 0 \\
0.05 & 0 \\
\end{bmatrix} \quad \begin{bmatrix}
0 & 0.73 \\
0 & 0 \\
0.66 & 0 \\
0.66 & 0 \\
0 & 0 \\
\end{bmatrix} \]
Coherence Pathways: Spin Echo

- Diagram shows non-zero states and evolution of states
- Perfect 180° pulses keep spins in low-order states
Example 2: Non-180° Spin Echo

• Ideal spin echo train gives simple RF rotations

• Now assume refocusing flip angles of 130°

• Compare RF rotations:

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\quad \text{180}_x^\circ
\]

\[
\begin{bmatrix}
0.18 & 0.82 & -0.77i \\
0.82 & 0.18 & 0.77i \\
0.38i & 0.38i & -0.64 \\
\end{bmatrix}
\quad \text{130}_x^\circ
\]

• Positive $F_n^+$ states remain because magnetization is not perfectly reversed, generating higher order states

• Many more coherence pathways (see next slide….)
Coherences: Non-180° Spin Echo

Only $F_0$ produces a signal... other $F_n$ states are perfectly dephased.
Example 3: Stimulated Echo Sequence

- We will follow this sequence through time
- Show which states are populated at each point
Example 3: Stimulated Echo Sequence

- Prior to the first pulse, all spins lie along $M_z$
- This is $Z_0=1$

![Diagram showing stimulated echo sequence with 60° pulses and 0° Z0 orientation]

![Voxel dimension graph with Mz and Z0 axes]
Stimulated Echo: Excitation: $F_0^+$

- After a 60° pulse, transverse spins are aligned along $M_y$ ($F_0^+ = \sin 60°$)

- One half ($\cos 60°$) of the magnetization is still represented by $Z_0$

*Note that we show the “voxel dimension” along different axis for Z and F states*
One Gradient Cycle

- The gradient “twists” the spins represented by $F^+_0$
- We call this state $F^+_1$, where the 1 indicates one cycle of phase ($F^+_1 = 0.86$)
- The spins represented by $Z_0$ are unaffected
Another Excitation (60°)

- The $Z_0$ magnetization is again split to $F^+_0$ and $Z_0$
- The $F^+_1$ magnetization is split *three* ways, to $F^+_1$, $F^-_1$ (reverse twisted) and $Z_1$
Another Gradient Cycle

- The $F^-_1$ state is refocused to $F^-_0$ or $F^+_0$
- The $F^+_0$ and $F^+_1$ states become $F^+_1$ and $F^+_2$
- The $Z$ states are all unaffected
- The process continues…!

\[ \text{Voxel Dimension} \]

\[ \text{Voxel Dimension} \]

\[ \text{Voxel Dimension} \]

\[ \text{Voxel Dimension} \]
Example 3: Coherence Pathways

- The stimulated echo sequence coherence diagram is shown below.
- Compare with F and Z states on prior slide (location of arrow).

The stimulated echo sequence coherence diagram is shown below.

Compare with F and Z states on prior slide (location of arrow).
Summary of Sequence Examples

• 90° and 180° RF pulses “swap” states
• Generally RF pulses “mix” states of order n
• Gradient pulses transition $F^+_n$ to $F^+_{n+p}$ and $F^-_n$ to $F^-_{n-p}$
• Coherence diagrams show progression through F and/or Z states to echo formation
• Signal calculation examples actually quantify the population of each state
Matlab Formulations

- Single matrix called “P” or “FZ” (for example)
- Rows are $F^+_n$, $F^-_n$ and $Z_n$ coefficients, Column each $n$
- RF, Relaxation are just matrix multiplications
- Gradients are shifts

\[ P = \begin{bmatrix} F_0 & F_1 & F_2 & \cdots & 0 \\ F^*_0 & F_{-1} & F_{-2} & \cdots & 0 \\ Z_0 & Z_1 & Z_2 & \cdots & 0 \end{bmatrix} \]
Matlab Formulations

• EPG simulations can be easily built-up using modular functions:
  (bmr.stanford.edu/epg)

– Transition functions:
  • epg_RF.m     Applies RF to Q matrix
  • epg_grad.m   Applies gradient to Q matrix
  • epg_grelax.m Gradient, relaxation and diffusion
  • epg_gdiff.m  Include diffusion effects

– Transformation to/from \((M_x, M_y, M_z)\):
  • epg_spins2FZ.m Convert M vectors to F,Z state matrix Q
  • epg_FZ2spins.m Convert F,Z state matrix Q to M vectors
Stimulated-Echo Example

- Simulate 3 Steps: RF, gradient and relaxation
- Sample Matlab code:

```matlab
function [S,Q] = epg_stim(flips)
Q = [0 0 1]';
for k=1:length(flips)
    Q = epg_rf(Q,flips(k)*pi/180,pi/2);
    Q = epg_grelax(Q,1,.2,0,1,0,1);
end;
S = Q(1,1);
```

- RF pulse: % RF pulse
- Gradient/Relax: % Gradient/Relax
- Signal from F0: % Signal from F0
Stimulated Echo Example (Cont)

Calculate signal vs flip angle:  
$fplot(epg_stim([x, x, x],[0, 120]))$
Diffusion

• Diffusion weighting can easily be applied

• Attenuation greater with \( n \) for both \( F_n \) and \( Z_n \)

• Requires physical “\( 2\pi \)” gradient twist \( k \) (\( m^{-1} \))

• If gradient is played, \( \Delta k = k \), otherwise \( \Delta k = 0 \)

\[
\begin{align*}
  b_n(k, \Delta k) &= \left[ \left( nk + \frac{\Delta k}{2} \right)^2 + \left( \frac{\Delta k^2}{12} \right) \right]^T \\
  F_n' &= F_n e^{-b_n(k, \Delta k)D} \\
  Z_n' &= Z_n e^{-b_n(k, 0)D}
\end{align*}
\]
Summary

- $F_n, Z_n$ basis represents many spins in a voxel
- MR operations on $F_n, Z_n$ states are simple
- Coherence diagrams show which states are non-zero
- Signal at any time is $F_0$. Other states are dephased

- Matrix formulation allows easy Matlab simulations:
  - Construct sequence of RF, gradient, relaxation, diffusion
  - Transient and steady state signals by looping
  - Can simulate multiple gradient directions
  - Can also simulate multiple diffusion directions (Weigel 2010)