Lecture-03B — Bloch Equation Matrix Simulations
Sequence Simulation Examples using Matrix Operations

Brian Hargreaves
bah@stanford.edu
Learning Objectives

• Use matrix operations for *analytic* calculation of simple signals
• Understand matrix representation of steady-states and transients
• Explain the hard-pulse approximation
• Build numerical simulations over time and space
Sequence Operations: Review

• Sequence blocks are simple matrix operations:
  
  – RF Excitation \[ M' = R_{\phi}(\alpha, t)M \]
  
  – Gradient-induced Rotation \[ M' = R_z(\gamma \vec{G}(t) \cdot \vec{r} + \gamma \Delta B_0)M \]
  
  – Relaxation \[ M' = A(\tau, T_1, T_2)M + B(\tau, T_1, T_2) \]

• Multiple spins, at different positions, can be simulated with loops
Example: Excitation/Recovery

\[ M_2 = R_y(-\alpha)M_1 = \begin{bmatrix} \cos{\alpha} & 0 & \sin{\alpha} \\ 0 & 1 & 0 \\ -\sin{\alpha} & 0 & \cos{\alpha} \end{bmatrix} M_1 \]

\[ M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_1 \end{bmatrix} M_2 + \begin{bmatrix} 0 \\ 0 \\ 1 - E_1 \end{bmatrix} \]

\[ M_1 = E_1 \cos{\alpha} M_1 + (1 - E_1) \]

\[ M_1 = \frac{1 - E_1}{E_1 \cos{\alpha}} \]

Neglect residual transverse magnetization

Common “steady-state” example from EE369B: (Fast T1-weighted MRI)
Overlapping RF/Gradients?

- z rotations and relaxation commute
- RF rotations do not commute with others
- **Hard-Pulse Approximation:**
  - Small rotations can be applied sequentially
  - Break pulses into very short-time segments
  - Basis for
    - Shinnar-Le Roux (SLR) pulse design
    - Variable Rate Selective Excitation “VERSE”
- Alternatively, calculate arbitrary rotations
- Add relaxation for full Bloch simulator

\[
\theta = \int_0^t \gamma B(\tau) d\tau
\]

\[
(\vec{M} \times \gamma \vec{B}_{net})
\]
Question 1

What are the equivalent rotations of $R_z(45°)R_x(45°)$ and of $R_x(45°)R_z(45°)$?

A. $R_\phi(\alpha)$ where $(\phi, \alpha) = 45°, 45°$
B. $R_\phi(\alpha)$ where $(\phi, \alpha) = 45°, 45°$
C. $R_\phi(\alpha)$ where $(\phi, \alpha) = 0°, 45°$ then $45°, 45°$
D. None of these

D. The axis of rotation is not in the transverse plane. A is also wrong because they cannot be equal. B and C are wrong if you just try starting with $M = \begin{bmatrix} 1 & 0 & 0 \\ T \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
Steady States with 3 Components

- Propagation over 1 TR: \( M_{n+1} = AM_n + B \)
- Steady State: \( M_{n+1} = M_n \)
- Combine: \( M_{ss} = AM_{ss} + B = (I-A)^{-1}B \)
  - If there is relaxation, there is a steady state
  - (eigenvalues of A are less than one in magnitude)
Question 2

If a sequence consists of a rotation about $Mx$ of 30°, and relaxation, what is the steady state, starting at equilibrium?

A. $M = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

B. $M = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

C. $M = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}$

D. $M = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

E. None of these

Rad229
Lec-03B
Slide-9

Rad229
Lec-03B
Slide-9
Transients

• Propagation over 1 TR: \( M_{n+1} = AM_n + B \)
• Steady State: \( M_{ss} = AM_{ss} + B \)
• Transient: \( M_{n+1} - M_{ss} = A(M_n - M_{ss}) = A^{n+1} (M_0 - M_{ss}) \)

• Eigenvector decomposition: \( A = V \Lambda V^{-1} \)
• Write: \( M_{n+1} - M_{ss} = V \Lambda^{n+1} V^{-1} (M_0 - M_{ss}) \)
Example: Free Precession

- Consider relaxation and precession over $\tau$:
  $$M' = R(E_1, E_2)R_z(\theta)M + B(E_1)$$

- Eigen-decomposition of $A$ is:
  $$\Lambda = \begin{bmatrix} E_2e^{i\theta} & 0 & 0 \\ 0 & E_2e^{-i\theta} & 0 \\ 0 & 0 & E_1 \end{bmatrix}$$
  $$V = \begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ \sqrt{2} & -i\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $\Lambda^k$ is rotation and attenuation
- $V$ has vectors that span the $M_x$-$M_y$ plane, and along $M_z$
- $M_{ss}$ is just $[0 \ 0 \ 1]^T$,
- $V^{-1}(M_0 - M_{ss})$ extracts $M_{xy}$ and $M_z$
- If $M_0 = [1 \ 0 \ 0]^T$ then $V^{-1}(M_0 - M_{ss}) = V^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -1 \end{bmatrix}$
- More interesting cases...!
Short-TR IR Signal

- Inversion Recovery Sequence:
  - TR = 1s, TI = 0.5s, TE=50ms
  - What is the signal for T₁=0.5s, T₂=100ms?

  - “Operations”
    - \( M_{180} = R_x(180)M_{TR} \)
    - \( M_{90} = R_x(90)E(0.5s)M_{180} \)
    - \( M_{TE} = E(0.05s)M_{90} \)
    - \( M_{TR} = E(0.45s = 1-0.5-0.05s)M_{TE} \)

  >> A1 = diag([E2a E2a E1a]) * Rx(180);
  >> B1 = [0;0;1-E1a];
  >> A2 = diag([E2b E2b E1b]) * Rx(90);
  >> B2 = [0;0;1-E1b];
  >> A3 = diag([E2c E2c E1c]);
  >> B3 = [0;0;1-E1c];
  >> A = A2*A1*A3;
  >> B = B2+A2*(B1+A1*B3);
  >> M = inv(eye(3)-A)*B
      [ 0; 0.2424; 0.0952 ]
Compact Simulation: abprop.m

- Propagate spins through series of A,B matrices
- Compact way to simulate sequences

```matlab
function [A,B,mss] = abprop(A1,B1,A2,B2,A3,B3,...)
If mss is provided, the steady-state is calculated.
If an Ai is 3x4, then it is assumed to be [Ai Bi]
If a Bi vector is omitted (the next argument is 3x3 or 3x4, it is assumed to be zero.
```

- `abprop(A1,B1, [A2 B2], A3, A4)` *(Here B3 = B4 = 0)*
Question 3

If you have a matrix, vector set $A, B$ that represents $N$ identical operations, can you uniquely determine $A_k$ and $B_k$ for each operation?

A. Yes, always
B. No, never
C. Sometimes, but multiple solutions for rotations
D. None of these

C. The $A_k$ matrix is simply the $N^{\text{th}}$ root of $A$, and $B_k$ can be solved (see absplit.m). Note if $A$ represents 180° rotation and $N=3$, then $A_k$ could represent rotations of +60° or -60° or other values.
Inversion Recovery Sequence:
- $TR = 1s$, $TI = 0.5s$, $TE=50ms$
- What is the signal for $T_1=0.5s$, $T_2=100ms$?

- “Operations”
  - $M_{180} = R_x(180)M_{TR}$
  - $M_{90} = R_x(90)E(0.5s)M_{180}$
  - $M_{TE} = E(0.05s)M_{90}$
  - $M_{TR} = E(0.45s = 1-0.5-0.05s)M_{TE}$

$$\gg [A,B,Mss] = abprop( ... \relax(TR-TE-TI,T1,T2,1), ... \xrot(180), ... \relax(TI,T1,T2,1), ... \xrot(90), ... \relax(TE,T1,T2,1));$$

$$\gg Mss \ [ 0; \ 0.2424; \ 0.0952 \ ]$$
Example: RF Pulse

- Simulation of RF/Gradient:
  - Time x Bandwidth 4
  - 4 cycles over slice
  - Uses complex M vector for each z position

Core Loop:

```matlab
for k=1:Nrf
    R = T*yrot(rf(k))*inv(T);  % Rotation from RF
    Mc = phmult.**(R*Mc);      % Apply RF and gradient rotations.
    Mxy = Txy*real(mc2mr(Mc)); % Extract only Mxy for plot
end
```

lec3_07.m
Example

- 90° Excitation pulse
  - Time samples of 4μs
  - 3 sinc cycles
  - 2ms duration
  - Area of 5.9 μT*ms
  - BW ~ 3 kHz
  - 2.3 mT/m gradient (1kHz/cm)
Simulation

- Loop over z
  - Define $R_z$
  - Loop over t
    - $M' = R_z R_y(t) M$
- Plot M over time and space
Same Example, with Off-Resonance

- 90 Excitation pulse
- BW ~ 3 kHz
- 2.3 mT/m gradient (1kHz/cm)
- 2kHz off-resonance??
Excitation Recovery (Real Pulse)

- Simulate full pulse and position
- Perfect spoiling ("keep only $M_z$" matrix)
- Matrix propagation to calculate steady-state at each position: $E() = \text{relaxation}$
  - End of RF to TR: Spoil, $E(\text{TR}-T_{RF})$
  - Over RF: $[E(\tau)R_z(\gamma Gzt) R_y(t,\tau)]$ at each interval
Example: Adiabatic Pulse

- Simulate complex $B_1$
- Track $M$ and $B_{\text{eff}}$
- Plot...

$[b1,freq,phase] = \text{adiabatic}(.15,1000,1000,0.008,.000004)$
Excitation and Gradient

- Can do this with real or complex M
  - Excite
  - Relaxation / gradient:

Core Loop:

```matlab
for k=1:200 % Start/end just after RF
    Mc = A*Mc+B; % Relaxation
    Mc = phmult.*Mc; % Gradient twist
    Mc = Rex * Mc; % Excitation
end;
plotm(mc2mr(Mc),0.5);
```

α=30° TR = 10ms TR = 10ms α=30° α=30° α=30° T1=1s, T2=0.2s
Question 4

In the last example (excitation + gradient), is there a signal immediately before the RF?

A. No
B. Yes, substantial
C. Yes, but very small
D. None of these

Rad229
Lec-03B
Slide-23
Summary

- Matrix steady-state is easily solved: $M_{ss} = (I-A)^{-1}B$
- Transients include rotation and attenuation
- Hard-pulse approximation for RF + precession
- Extend to full Bloch simulation
- Brute-force space and time dimension simulations
How do you put this together to efficiently simulate multiple spins?