Lecture-9A — Gradient-Echo Sequences
Mathematics of Balanced SSFP (bSSFP)

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Learning Objectives

• Describe a matrix formulation of the bSSFP signal
• Describe a geometrical formulation of the signal
• Explain transient signals mathematically
Matrix Derivation of bSSFP Signal

• **Sequence:** $E_{\text{Relaxation}} - R_z(180^\circ) - R_y(\alpha)$
  - Note $R_z(180^\circ)$ lets us do this over only 1TR

• $m(0) = R_y(\alpha) \cdot R_z(180^\circ) \cdot E \cdot m(0) + R_y(\alpha) \cdot [0;0;1-E_1]$
  - Note: no $M_y$

\[
\begin{bmatrix}
1 + E_2 \cos \alpha & 0 & -E_1 \sin \alpha \\
0 & 1 + E_2 & 0 \\
-E_2 \sin \alpha & 0 & 1 - E_1 \cos \alpha
\end{bmatrix} m(0) = \begin{bmatrix}
\sin \alpha(1 - E_1) \\
0 \\
\cos \alpha(1 - E_1)
\end{bmatrix}
\]

Some algebra...

\[
m(0) = \frac{1 - E_1}{1 + \cos \alpha(E_2 - E_1) - E_1E_2} \begin{bmatrix}
\sin \alpha \\
0 \\
E_2 + \cos \alpha
\end{bmatrix}
\]

The matrix derivation gives the exact steady-state vs flip angle for $T_1$ and $T_2$. 
Question 1: Steady-state Elevation Angle

The angle from the $m_z$ axis of the steady-state magnetization, $m(0)$, is tilted by $\alpha/2$ (good!).

This matrix derivation passes some basic checks.
Question 2: Short-$T_2$?

If $E_2 = 0$, same as excitation-recovery (good!)

$$m(0) = 1 - E_1 + \cos \alpha (E_2 - E_1) - E_1 E_2 \sin \alpha$$

RF TR 60° 60° 60°
Matrix Derivation of bSSFP Signal with off-resonance

- **Sequence:** \( E_{\text{Relaxation}} - R_z(180^\circ + \phi) - R_{-y}(\alpha) \)
  - Note \( R_z(180^\circ) \) still lets us do this over only 1TR

\[
m(0) = \frac{ae^{i\phi} + b}{c \cos \phi + d}
\]

\[
a = -(1 - E_1)E_2 \sin \alpha \\
b = (1 - E_1)\sin \alpha \\
c = E_2(E_1 - 1)(1 + \cos \alpha) \\
d = 1 - E_1 \cos \alpha - (E_1 - \cos \alpha)E_2^2
\]

\[
E_{1,2} = e^{-T_{1,2}/TR}
\]

The equation is for signal after the RF pulse, with off-resonance precession.  lec9_01.m
Matrix Signal: Consideration of Precession and $T_2$-decay

- Derived on-resonance, but immediately after RF
  - Magnetization is refocused at $TE = TR/2$
  - Magnetization has linear phase at $TE = 0$
- (Small) $T_2$ decay to $TE$ of $e^{-TE/T_2}$

The signal immediately after the RF has linear phase, and decays slightly over the TR.
“Ellipsoid” Signal Derivation

- Start with ellipsoid:
  \[
  \left( M_z - \frac{M_0}{2} \right)^2 + \frac{M_x^2 + M_y^2}{T_2/T_1} = \left( \frac{M_0}{2} \right)^2
  \]
- Substitute \( M_z = M \cos(\beta/2) \) and \( M_{xy} = M \sin(\beta/2) \):
  \[
  \left[ M \cos(\beta/2) - \frac{M_0}{2} \right]^2 + \frac{M^2 \sin^2(\beta/2)}{T_2/T_1} = \left[ \frac{M_0}{2} \right]^2
  \]
- Multiply out, bring up \( T_2/T_1 \):
  \[
  M^2 \cos^2(\beta/2) + (T_1/T_2)M^2 \sin^2(\beta/2) = MM_0 \cos(\beta/2)
  \]
- Divide out \( M \), rearrange and multiply by \( \sin(\beta/2) \):
  \[
  M \sin(\beta/2) = \frac{M_0 \cos(\beta/2) \sin(\beta/2)}{\cos^2(\beta/2) + (T_1/T_2)M^2 \sin^2(\beta/2)}
  \]
- Divide terms along denominator by numerator:
  \[
  M_{xy} = \frac{M_0}{\cot(\beta/2) + (T_1/T_2) \tan(\beta/2)}
  \]

- Signal drops with increasing \( T_1/T_2 \)
- At \( \beta = 180^\circ \) signal is 0.
- At \( \beta = 90^\circ, T_1 = T_2, S = M_0/2 \)
Compare Ellipsoid Signal to Matrix Derivation:

• Matrix ($\alpha$ replaced with $\beta$):

$$m(0) = \frac{M_0(1 - E_1)}{1 + \cos \beta(E_2 - E_1) - E_1 E_2} \begin{bmatrix} \sin \beta \\ 0 \\ E_2 + \cos \beta \end{bmatrix}$$

• Approximate $E_{1,2} = e^{-TR/T_{1,2}} \approx 1 - TR/T_{1,2}$ and neglect $TR^2/(T_1 T_2)$. Substitute and divide out $TR/T_1$:

$$M_{xy} = \frac{M_0 \sin \beta}{(1 + \cos \beta) + (T_1/T_2)(1 - \cos \beta)}$$

• Identities $\sin(\beta) = 2 \sin(\beta/2)\cos(\beta/2)$, $1 + \cos(\beta) = 2 \cos^2(\beta/2)$ and $1 - \cos(\beta) = 2 \sin^2(\beta/2)$:

$$M_{xy} = \frac{M_0 \sin(\alpha/2)\cos(\alpha/2)}{\cos^2(\alpha/2) + (T_1/T_2)\sin^2(\alpha/2)}$$

• Divide out $\sin(\beta/2)\cos(\beta/2)$:

$$M_{xy} = \frac{M_0}{\cot(\beta/2) + (T_1/T_2)\tan(\beta/2)}$$

Ellipsoidal Derivation!
Balanced SSFP: Transients

- Transient paths differ based on initial state

The transient path depends on the initial condition
Transients: Matrix Solutions

Apply 3x3 matrix scheme:
\[ M_{k+1} = A M_k + B \]  \hspace{1cm} [1]

In steady-state:
\[ M_{ss} = A M_{ss} + B \]  \hspace{1cm} [2]
\[ M_{k+1} - M_{ss} = A(M_k - M_{ss}) \]  \hspace{1cm} [1-2]

Consider Eigenvector Decomposition:
\[ A = V \Lambda V^{-1} \]
\[ M_k - M_{ss} = V \Lambda^k \left[ V^{-1}(M_0 - M_{ss}) \right] \]

At least one eigenvector/value is real.
Others often oscillatory and die out in steady state

Note in [2] \( A \) is mostly rotation, \( B \) is small, so \( M_{ss} \) lies almost along the real-valued eigenvector

The transient can be characterized by the difference from initial state to steady state. Jaynes 1955
Balanced SSFP: Transient (no precession)

A back-and-forth excitation is the same as a constant rotation with 180° precession.
Transient: Off-resonance

The initial component parallel to the steady state decays smoothly, while the orthogonal component oscillates.
Transients (General)

• Generally include two components:
  • **Smooth** exponential (useful!)
  • **Oscillatory** (problematic)

• Smooth transient is along steady-state direction
• Manipulate to steady-state direction to avoid oscillations
Half-TR, $\alpha/2$ Setup

- First RF pulse has half-amplitude
- Pulse applied TR/2 before next RF pulse
- (More complicated schemes exist)

For on-resonance magnetization, places $M - M_{ss}$ along steady-state. Deimling and Heid, 1994
bSSFP Direction: Some Intuition

- Magnetization aligns to rotation axis
- Rotation $\theta$ includes RF phase-increment
- As $\theta$ approaches 0°, axis is transverse, signal dies out
- Negative $\theta$ means steady state on -x

The steady-state lies close to the “effective” rotation axis
A matrix derivation of the bSSFP signal in 2D can be done using an alternating reference frame.

A geometric signal derivation arises from the ellipsoid model.

Transients can be described by a matrix exponential:
- Along steady state, transient is smooth
- Perpendicular component is oscillatory

bSSFP steady state can be considered as the “effective” rotation axis.
How does this relate to other gradient-echo sequences?