RAD229: Final Exam – 2014/2015 - SOLUTIONS

You will have 3 hours to complete this Exam

Solutions are given in Blue.

In some cases, different interpretations may have led to different, but reasonable answers, and these may not all be reflected in this solution set.

General Instructions:
1. There are 10 questions and 12 pages total.
2. You may use notes including lectures, homework, solutions or Matlab script text on the course website, and you may use an electronic device for reading such material.
3. You may not use Matlab or the Internet, or other calculating devices of any sort.
4. Please answer questions on the exam, and show your final answer clearly.
5. Show your reasoning and work, as this will often earn you partial points.
6. You may request more paper if needed.
7. For this exam, please a gyromagnetic ratio value \( \gamma/2\pi = 40.0 \text{ MHz/T} \)
8. The following plot, which we have used repeatedly, is given for the above \( \gamma \) value and gradients with maximum amplitude 50mT/m and maximum slew rate 200mT/m/ms:

The EPG RF rotation matrix \( R(\alpha,\phi) \) is

\[
\begin{bmatrix}
\cos^2(\alpha/2) & e^{2i\phi}\sin^2(\alpha/2) & -ie^{i\phi}\sin\alpha \\
e^{-2i\phi}\sin^2(\alpha/2) & \cos^2(\alpha/2) & ie^{-i\phi}\sin\alpha \\
-i/2e^{-i\phi}\sin\alpha & i/2e^{i\phi}\sin\alpha & \cos\alpha
\end{bmatrix}
\]

Please do not proceed to the next page until the exam begins.
Short Answer Questions (15% of Points)

1. Describe, intuitively and briefly, the effect on an image of the following k-space modulation effects:

   a. Linear phase modulation by N cycles over the full k-space in the $k_x$ direction

      *Pixel shift of N pixels in x*

   b. A unit step discontinuity (along $k_y=0$) modulation of k-space magnitude or phase

      *Ringing in the direction perpendicular to the step. Depending on if you offset the step (no offset if -0.5 to 0.5) then you add a delta-function to the PSF. Since that’s unspecified, ringing is the only required answer.*

   c. Smooth decrease of magnitude from k-space center outward

      *Blurring, due to attenuation of higher spatial frequencies.*

   d. Smooth change of phase from k-space center outward

      *Blurring, due to higher spatial frequencies out of phase with lower spatial frequencies.*
2. Given the signal contours below (vs T₁ and T₂), what is the type of contrast, and describe (briefly) spin-echo sequences and parameters (TR, TE) and/or variations studied that could produce this contrast. *You do not need the parameter numbers to be exact!*

![Signal Contours](image)

**a.** T₂ contrast. *Can be achieved with spin echo sequence with long TR and moderate or long TE, say TR/TE = 4000/100ms*

![Signal Contours](image)

**b.** T₁ contrast. *Can be achieved with spin echo sequence with short TR and short TE, say TR/TE = 500/10 ms*

![Signal Contours](image)

**c.** T₂/T₁ contrast. *Best achieved with spin echo sequence with tipup pulse (driven equilibrium, see HW#1) with short-to-moderate TR and reasonable TE (or echo train length). Acceptable is Spin Echo with “mixed” weighting, short TR and longer TE, say TR/TE=400/100ms, but the former would give you much higher signal.*
3. **Fat/Water Separation:** Consider a 2-point Dixon sequence at 3T where the fat/water chemical shift is 400 Hz (slight simplification from actual 440 Hz). The reconstruction does NOT correct any phase effects due to off-resonance or effects from relaxation.

The reconstruction is

\[
\text{Fat}(F) = 0.5 \left[ S(TE_2) - S(TE_1) \right] \quad \text{Water}(W) = 0.5 \left[ S(TE_2) + S(TE_1) \right]
\]

a. What are appropriate (minimal) echo times for the acquisition for a gradient-echo sequence?

**In-phase TE is 1/400Hz or 2.5ms. Out-of-phase is half of this, 1.25ms**

b. If a voxel contains only water (amplitude 1), and is actually 200Hz off-resonance, what will the reconstructed water and fat images be? (Ignore relaxation).

**The signals at echo times will be 90° out of phase, so both the sum and difference are sqrt(2), and water/fat are both sqrt(2)/2.**

\[ W = 0.7, \ F = 0.7 \]

c. If a voxel contains only water and is on-resonance, what are the water and fat signals as a function of \( T_2 \) (Assume the same \( T_2 \) for fat and water)

\[ W = 0.5 \left[ \exp(-TE_1/T_2) + \exp(TE_2/T_2) \right] \]

\[ F = 0.5 \left[ -\exp(-TE_1/T_2) + \exp(TE_2/T_2) \right] \]

d. If the individual images at different echo times have SNR of 50 for water-only pixels, what will be the SNR of the water image be?

**W is a linear sum, so sqrt(2) benefit, so SNR = 70.7 or 50 sqrt(2).**
Medium Answer Questions (25% of Points)

4. **Projection-Reconstruction with Motion.** An object is moving *rigidly* in-plane with periodic “cardiac” motion described $\Delta x(t)$ and $\Delta y(t)$, which are both band-limited to 50Hz. The period is at least 50ms, but can vary over time by about 10%.

   a. If the object shift due to motion is $(\Delta x, \Delta y)$, what is the corresponding k-space modulation pattern?

$$\exp \left[ 2\pi i \left( \Delta x k_x + \Delta y k_y \right) \right] \quad (\text{Assuming parameters given in reciprocal units mm, mm}^{-1} \text{ etc.})$$

$$\exp \left[ 2\pi i \left( \Delta x k_x + \Delta y k_y \right) / \text{FOV} \right] \quad (\text{Assuming parameters given in samples})$$

   b. If no motion correction is performed and a golden-angle ordering is used, explain the impact on the image due to motion.

   *Generally the image will blur. Linear phases will minimally affect the area near the center of k-space, but will cause phase cancellation near the edges.*

   c. Using of projections (from $-k_{max}$ to $k_{max}$), what is the minimum number needed to determine the position relative to a complete static reference image of the object? Explain how you would measure position.

   *2 projections, ideally orthogonal, but actually any non-parallel, will let you determine position. Since you have a static reference you can extract the phase difference along the projections, fit the linear component, and this gives you the position along the projections, which gives you the overall 2D position.*

   d. Describe a projection angle order that ensures that you measure the position at a high frame rate, and describe this rate.

   *Acquire projections at 0°, 90°, a°, 90+a°, 2a°, 90+2a°, ...*

   *This gives position every 2TRs, or even a sliding-window can give you updates every TR.*

   *The golden-angle increment is close enough to 90° that just using 2 consecutive projections should give a good result here.*

   *(Note the challenge in practice is getting that reference image in part c!)*

   e. How many projections are needed to acquire a FOV of diameter 20cm with image resolution of 1mm, assuming gradient amplitude and sampling rate are not limitations?

   *This is basic 2DPR with full spokes. The Cartesian matrix here is 200x200, so instead of 200 ky lines, you will need 200 $(\pi/2)$ projections, or 314.*
5. **Diffusion:** You are imaging a species that has diffusive motion with a uniform (rather than Gaussian) distribution. The diffusion probability density function is \( p(\Delta x) = 1/X \) for \( |\Delta x| < X/2 \), where \( X = \sqrt{D \Delta t}/2 \), and \( D \) is the diffusion coefficient (mm\(^2\)/s, for example). Assume magnetization starts at equilibrium.

a. Using the sequence at right, where diffusion gradients have area \( GT \) (mT/m*ms, for example) but are instantaneous, write an expression for the signal at TE as a function of \( G, T, D, \Delta t, TE \) and \( T_2 \).

\[
\text{The distribution of phases is calculated over rect}(x/X) \text{ which means we integrate } \exp(-2\pi i x k_x) \text{ from } -X/2 \text{ to } X/2. \text{ This integral is precisely the Fourier transform of rect}(x/X) \text{ evaluated at } k_x = (\gamma/2\pi) GT, \text{ so } \text{sinc}(X k_x) \text{ or sinc } \left[ \frac{\gamma GT \sqrt{D \Delta t}}{4\pi} \right].
\]

\[
\text{It follows that the signal, including } T_2 \text{ relaxation is } \text{sinc } \left[ \frac{\gamma GT \sqrt{D \Delta t}}{4\pi} \right] \exp(-TE/T_2).
\]
6. **Sampling and SNR:** A researcher is sampling an image using full-k-space 2D Cartesian sampling, sampling in both $k_x$ and $k_y$ over an extent from $-k_{max}$ to $k_{max}$. The researcher wants to improve the SNR.

   a. What is the impact on (i) SNR, (ii) scan time and (iii) SNR efficiency of increasing (only) the FOV in the phase encode direction by a factor of 1.5?

      i) $\sqrt{1.5}$ increase.  (ii) 1.5x increase.  (iii) no change since SNR/$\sqrt{\text{scan}} = 1$.

   b. The researcher instead chooses to sample the central 50% of the $k_y$ lines twice, and simply adds the acquired signal at each of these lines before performing the reconstruction. Write an expression for the PSF in the y direction, i.e. $p(y)$, and explain the effect of extra sampling on the PSF.

      *The basic PSF is $\text{sinc}(2k_{max}y)$. For the central resampled region, the extent is smaller so (relative) PSF is $0.5\text{sinc}(k_{max}y)$. By superposition, the PSF is $\text{sinc}(2k_{max}y) + 0.5\text{sinc}(k_{max}y)$, which results in broadening.*

   c. What step must be added to the reconstruction in order to not affect the point-spread function?

      *Density compensation must be included, so averaging (vs adding) the twice-sampled signals.*

   d. Assuming (c) is done correctly, what is the impact on noise spectral distribution (short, qualitative explanation).

      *The noise is no longer white. It will have more emphasized high-spatial frequencies in the y direction.*

   e. Does the SNR efficiency change with the sampling in b and the reconstruction in c, and if so, by what amount?

      *Yes. The sampled area in k-space is the same ($A$ in our $\eta$ calculation). The average density is $(1+2)(1/2) = 3/2$. The average inverse-density is $(1+1/2)(1/2) = 3/4$. The overall change (decrease) in efficiency is $1/ \sqrt{(3/2)(3/4)}$ or $\sqrt{8/9}$.*

   f. The researcher has a new idea and instead of (a), increases the FOV in the readout direction by a factor of 1.5, while also increasing the sampling rate (and bandwidth) by a factor of 1.5. How does the SNR change?

      *Voxel volume and readout duration didn’t change, so SNR didn’t change. FOV comes for free in the readout direction!*
7. **CPMG Sequence Design:** Consider a non-selective (3D) CPMG sequence with 180° refocusing pulses where you wish to image with 1mm resolution and a FOV of 40cm in the readout direction in a minimum time. The system gradients are maximum amplitude 50mT/m, maximum slew rate 200mT/m/ms (250µs rise time), and the receiver can sample with a maximum half-bandwidth of 100kHz. The peak B₁ amplitude is 15µT.

   a. What is the minimum RF pulse duration for 180° pulses (rectangular pulses)?

   \[(\gamma/2\pi)B_1T = 0.5 \text{ cycles}\]

   \[T = 0.5 / (40 \times 0.015) = 0.5/0.6 = 0.833 \text{ ms}\]

   b. If the z-resolution is 2mm, and crusher pulses (on Gₓ) must dephase 1 cycle over a voxel, what is the duration of each crusher pulse?

   \[k_z \text{ for crushers } = 1/2mm = 5cm^{-1}. \text{ Using the table, this gives } 0.5\text{ms}\]
   \[\text{Note that this is } 2x \text{ the rise so the pulse is a perfect triangle to full scale.}\]

   c. What is the readout gradient amplitude (mT/m), and plateau duration (ms), assuming sampling only on flat portions of the gradient?

   Based on bandwidth, \(T_s = 5\mu s = 0.005\text{ms}\).
   \[\frac{\gamma}{2\pi}G_Ts = 1/0.4\text{m}\]
   \[G = 2.5\text{m-1} / .005\text{ms} / 40kHz/mT = 2.5/0.2 = 12.5 \text{ mT/m}\]
   \[\text{This is less than } G_{\text{max}} \text{ so makes sense to use. The duration is 400 samples, so } 400\times0.005\text{ms } = 2\text{ms}\]

   d. What is the minimum echo spacing (in ms)? Echo spacing is the time from one refocusing pulse to the next or (equivalently) from one spin echo to the next.

   \[\text{We must include the full readout plateau, full RF and 2 crushers. The readout ramps can overlap the crushers. Thus the answer is } 2\text{ms} + 2(0.5\text{ms}) + 0.833\text{ms } = 3.83\text{ms}\]

   e. Draw the pulse sequence (RF, Gₓ and Gz) from the 90° excitation to the second refocusing pulse that corresponds to the minimum echo spacing. Label all scales and axes.
8. **Spin Echo Trains.** For this question you are to design a CPMG spin-echo-train sequence (with crushers) with constant 100° refocusing pulses over all of the echo train (except the first pulse). Assume the magnetization starts at equilibrium ($M_0$) – you are just looking at dynamics from equilibrium here.

a. What should the flip angle of the first refocusing pulse (after the 90° excitation) be, and why?

$$90 + 100/2 = 140^\circ.$$ This helps to stabilize the echo train.

b. Draw the EPG coherence pathway diagram for the first 3 echoes

This is directly from the class notes. (Not all pathways lead to signal at echoes here)

![Echo Pathway Diagram]

---

<table>
<thead>
<tr>
<th>Pathway</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
<td>$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
</tr>
<tr>
<td>2)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
<td>$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
</tr>
<tr>
<td>3)</td>
<td>TE/2</td>
<td>$F_1$-140°-$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
<td>$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
</tr>
<tr>
<td>4)</td>
<td>TE/2</td>
<td>$F_1$-140°-$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
<td>$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
</tr>
<tr>
<td>5)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_3$</td>
<td>100°</td>
<td>$F_{-3}$</td>
<td>TE</td>
<td>$F_{-1}$</td>
<td>100°</td>
</tr>
</tbody>
</table>

---

c. How many distinct pathways lead to the observed signal at the 3rd spin echo? (List the pathways by the EPG states before/after each refocusing pulse.)

Five pathways described as follows, starting after the 90 pulse:

<table>
<thead>
<tr>
<th>Pathway</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
<th>Time</th>
<th>Sequence</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
<td>$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
</tr>
<tr>
<td>2)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
<td>$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
</tr>
<tr>
<td>3)</td>
<td>TE/2</td>
<td>$F_1$-140°-$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
<td>$F_{-1}$</td>
<td>TE</td>
<td>$F_1$</td>
<td>100°</td>
</tr>
<tr>
<td>4)</td>
<td>TE/2</td>
<td>$F_1$-140°-$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
<td>$Z_{-1}$</td>
<td>TE</td>
<td>$Z_1$</td>
<td>100°</td>
</tr>
<tr>
<td>5)</td>
<td>TE/2</td>
<td>$F_1$-140°-$F_{-1}$</td>
<td>TE</td>
<td>$F_3$</td>
<td>100°</td>
<td>$F_{-3}$</td>
<td>TE</td>
<td>$F_{-1}$</td>
<td>100°</td>
</tr>
</tbody>
</table>

d. Give an expression for the signal on echoes 1 and 2 as a function of TE, $T_1$ and $T_2$. You may include terms such as sin(90°) in this expression.

Here we use $E_2=\exp(-TE/T_2)$, and $E_1=\exp(-TE/T_1)$ and Assume $\varphi=0$ for RF for simplicity.

**Echo 1:** $S = E_2 \sin^2(70^\circ)$ – just the spin-echo using a 140° pulse, element 2,1 in the EPG RF rotation matrix.

**Echo 2:** Primary + one stimulated echo pathway.

For the primary spin echo pathway, the signal is just another spin echo from Echo 1, but with a different flip angle. For the stimulated echo, we use elements 3,1 and 2,3 of the EPG rotation matrix, and the pathways add constructively due to $i^*(-i)$. The transverse and longitudinal decay multiply the pathways:
\[ S_2 = E_2^2 \sin^2(70^\circ) \sin^2(50^\circ) + E_1 E_2 \sin(140^\circ) \sin(100^\circ)/2 \]
9. **Echo-Planar Imaging (EPI).** Consider a single-shot, bipolar 2D EPI imaging trajectory used to image the 2D object below, which has amplitude 1 for points within a radius of 75mm of the center and 0 elsewhere. This can be written as \( M(x,y) = \text{circ}(r/A) \) where \( A = 75\text{mm} \) and \( r = \sqrt{x^2 + y^2} \), and \((x,y)\) are image-space coordinates. The object is imaged with 2D EPI with a 25cm FOV and 1mm resolution in both \( x \) and \( y \) directions.

b. Assume that there is a constant phase difference between odd and even phase-encode lines \( (k_i) \) of \( \pi/4 \) (i.e. even lines are multiplied by \( \text{exp}(i\pi/8) \) and odd lines by \( \text{exp}(-i\pi/8) \). Give an expression for the resulting magnitude image, to within a constant scale factor. (writing \(|a+bi|\) is sufficient.)

Let \( A(x,y) = \text{M}(x,y) \) the primary signal and \( B(x,y) = \text{“N/2 Ghost”} = A(x,y+/-\text{FOV}/2) \) where +/- is chosen so that \(|y+/-\text{FOV}/2| < \text{FOV}/2\).

**Even Echoes:** \( s = 0.5[A \ \text{exp}(i\pi/8) + B \ \text{exp}(i\pi/8)] \)

**Odd Echoes:** \( s = 0.5[A \ \text{exp}(-i\pi/8) + B \ \text{exp}(-i\pi/8)] \)

**Net Image:** \( s = A\cos(\pi/8) + iB\sin(\pi/8) \)

c. Sketch the resulting magnitude image at right, labeling magnitudes (within a scale factor).

The primary image magnitude is \( \cos(\pi/8) \), while the replica is \( \sin(\pi/8) \), and since these are out-of-phase, where they intersect the magnitude is 1(!).

d. Now there is additionally a sampling delay two samples in k-space along the whole trajectory. (Effectively odd lines are shifted in one direction along readout by 1 sample, even lines shifted in the other direction by one sample.) Give an expression for the magnitude image. (writing as \(|a+bi|\) is sufficient.)

In the image we add linear phase \( \text{exp}(i2\pi x/\text{FOV}) \) and \( \text{exp}(-i2\pi x/\text{FOV}) \) to odd/even line images, which just is a phase on the cosine/sin.

**Net Image:** \( s = A\cos(\pi/8 + 2\pi x/\text{FOV}) + iB\sin(\pi/8 + 2\pi x/\text{FOV}) \)

e. Describe differences between the image resulting in (c) and the image in (a,b).

The images in c have amplitude varying along the readout, whereas in a,b the amplitude is not affected by the EPI phase error.
10. **Spiral Design**: You are asked to image with 1mm resolution and 12cm (diameter) FOV using Archimedean spiral imaging using a number of interleaves between 10 and 16. For the Archimedean spiral, \( k = A \theta \exp(i\theta) \), and \( k_x = \text{real}\{k\}, \ k_y = \text{imag}\{k\} \), where \( A \) is a constant.

a. What is the effect of off-resonance (briefly)? What is a good choice of number of interleaves to minimize this effect?

*Off resonance causes a blurring. To minimize this, we should maximize the number of interleaves, so choose 16.*

b. Is there any image quality benefit of reducing the number of interleaves slightly here? Explain?

*There may be a small benefit to using an odd number of interleaves, so 15. This makes the k-space trajectory a half-Fourier trajectory and some benefit may arise from reduced aliasing.*

c. Draw the desired k-space trajectory at right for a single interleaf of this trajectory, labeling scales.

*The trajectory must do 120/2 turns in 15 interleaves, so 4 turns per interleaf, with spacing of 1. 25cm\(^{-1}\) to reach a maximum k-space radius of 5cm\(^{-1}\), which corresponds to 1mm resolution.*

d. If we later decide that we need double the FOV, how can we use this same waveform to achieve our imaging goal?

*Simply double the number of interleaves to 30*

e. Write an equation for each of the gradients \( G_x(t) \) and \( G_y(t) \) if \( \theta(t) = Bt \).

\[
\begin{align*}
\mathbf{k} &= A \theta \exp(i\theta), \quad \mathbf{G} = d\mathbf{k}/dt / (\gamma/2\pi), \\
\mathbf{G} &= (\gamma/2\pi)d\theta/dt, \quad AB (t \exp(iBt)) = AB (1+iBt) \exp(iBt) \\
G_x &= (\gamma/2\pi) [AB \cos(Bt) - AB^2 t \sin(Bt)] \\
G_y &= (\gamma/2\pi) [AB \sin(Bt) + AB^2 t \cos(Bt)]
\end{align*}
\]

*Although it was not requested, \( A = N / (2\pi \text{FOV}) \)*

e. Describe what considerations must be made to address each of the following, or “none” if the system parameter does not affect the waveform design. Answer briefly.

i. Maximum gradient amplitude (\( G_{\text{max}} \)) \( |G| < G_{\text{max}} \)

*This affects speed along the trajectory, primarily later on in the trajectory.*

ii. Minimum data sampling interval (\( T_s \)) \( |G| < 1/\text{FOV} / (\gamma/2\pi) / T_s \)

*Similar to (i) but sometimes FOV is the limit. This does happen in practice!*

iii. Maximum gradient slew rate \( |dG/dt| < S_{\text{max}} \)

*This primarily affects the trajectory early on, when it is turning quickly.*