Problem 1

Let \((a_n)_{n \geq 1}\) be a sequence of numbers such that \(a_n \geq 0\) for all \(n\), and \(\lim_{n \to \infty} a_n = 0\). Consider random variables \(X_n \sim \mathcal{N}(0, a_n)\), not necessarily independent.

(i) Is there a random variable \(X_{\infty}\) such that \(X_n \xrightarrow{p} X_{\infty}\)? Justify your answer.

(ii) Is there a random variable \(X_{\infty}\) such that \(X_n \xrightarrow{d} X_{\infty}\)? Justify your answer.

(iii) Let \(q \geq 1\). Is there a random variable \(X_{\infty}\) such that \(X_n \xrightarrow{L^q} X_{\infty}\)? Justify your answer.

(iv) Consider the case \(a_n = 1/n\). Is there \(X_{\infty}\) such that \(X_n \xrightarrow{a.s.} X_{\infty}\)? Justify your answer.

(v) Repeat point (iv) for \(X_n\) independent and \(a_n = 1/\log \log n\).

Problem 2

Find random variables \(X\) and \((X_n)_{n \geq 1}\) such that \(\lim_{n \to \infty} \mathbb{E}[|X_n - X|^2] = 0\), but \(\mathbb{E}[X_n^2] = \infty\) for all \(n\).

Problem 3

In defining a renewal process we suppose that \(F(\infty)\), the probability that an interarrival time is finite, equals 1. If \(F(\infty) < 1\), then after each renewal there is a positive probability \(1 - F(\infty)\) that there will be no further renewals. Argue that when \(F(\infty) < 1\) the total number of renewals, call it \(N(\infty)\), is such that \(1 + N(\infty)\) has a geometric distribution with mean \(1/(1 - F(\infty))\).

Problem 4

Express in words what the random variable \(X_{N(t)+1}\) represents (Hint: It is the length of which renewal interval?) Show that

\[ P\{X_{N(t)+1} \geq x\} \geq \bar{F}(x) \]

Compute the above exactly when \(F(x) = 1 - e^{-\lambda x}\).
Problem 5

Prove the renewal equation

\[ m(t) = F(t) + \int_0^t m(t - x) dF(x). \]