Problem 1

Let \((a_n)_{n \geq 1}\) be a sequence of numbers such that \(a_n \geq 0\) for all \(n\), and \(\lim_{n \to \infty} a_n = 0\). Consider random variables \(X_n \sim N(0, a_n)\), not necessarily independent.

(i) Is there a random variable \(X_\infty\) such that \(X_n \xrightarrow{p} X_\infty\)? Justify your answer.

(ii) Is there a random variable \(X_\infty\) such that \(X_n \xrightarrow{d} X_\infty\)? Justify your answer.

(iii) Let \(q \geq 1\). Is there a random variable \(X_\infty\) such that \(X_n \xrightarrow{L^q} X_\infty\)? Justify your answer.

(iv) Consider the case \(a_n = 1/n\). Is there \(X_\infty\) such that \(X_n \xrightarrow{a.s.} X_\infty\)? Justify your answer.

(v) Repeat point (iv) for \(X_n\) independent and \(a_n = 1/\log \log n\).

Solution.

(i) Take \(X_\infty = 0\). For any \(\varepsilon > 0\), we have (Chebyshev’s inequality)

\[
P(|X_n| \geq \varepsilon) \leq \frac{E[X_n^2]}{\varepsilon^2} = \frac{a_n}{\varepsilon^2} \to 0 \Rightarrow X_n \xrightarrow{p} 0.
\]

(ii) \(X_n \xrightarrow{p} 0\) implies \(X_n \xrightarrow{d} 0\).

(iii) Since \(X_n \sim N(0, a_n)\), we have \(X_n \xrightarrow{d} a_n^{1/2}Z\) where \(Z \sim N(0, 1)\), thus leading to

\[
E[|X_n|^q] = a_n^{q/2}E[|Z|^q] \to 0 \text{ as } n \to \infty.
\]

Hence, \(X_n \xrightarrow{L^q} 0\).

(iv) We will prove that \(X_n \xrightarrow{a.s.} 0\). Let \(\varepsilon_n = n^{-\alpha}\) for some \(\alpha \in (0, 1/4)\), then we have

\[
\sum_{n=1}^{\infty} P(|X_n| \geq \varepsilon_n) \leq \sum_{n=1}^{\infty} \frac{E[|X_n|^4]}{\varepsilon_n^4} = \sum_{n=1}^{\infty} \frac{3}{n^2 \varepsilon_n^4} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 - 4\alpha} < \infty.
\]

According to Borel-Cantelli Lemma, we have \(P(|X_n| \geq \varepsilon_n, \text{ i.o.}) = 0\). This further implies that with probability one, \(|X_n| \leq \varepsilon_n\) for all large enough \(n\). Hence, \(X_n \xrightarrow{a.s.} 0\) almost surely.
(v) We prove that with probability one, $X_n$ does not converge to 0. To this end, we fix an $\varepsilon > 0$ and show that
\[
\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq \varepsilon) = \infty.
\]

Since the $X_n$’s are mutually independent, applying Borel-Cantelli Lemma yields that $\mathbb{P}(|X_n| \geq \varepsilon, \text{ i.o.}) = 1$, which further implies the desired result.

Now we prove our claim. According to Mills’ ratio, we have
\[
\mathbb{P}(|X_n| \geq \varepsilon) = 2\mathbb{P}(X_n \geq \varepsilon) = 2\mathbb{P}\left(Z \geq \frac{\varepsilon}{\sqrt{a_n}}\right) \sim \sqrt{\frac{2}{\pi}} \frac{\sqrt{a_n}}{\varepsilon} \exp\left(-\frac{\varepsilon^2}{2a_n}\right)
\]
\[
= \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon \sqrt{\log \log n}} (\log n)^{-\varepsilon^2/2} \gg \frac{1}{n} \quad \text{for large } n.
\]

As a consequence, we have $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq \varepsilon) = \infty$.

**Problem 2**

Find random variables $X$ and $(X_n)_{n \geq 1}$ such that $\lim_{n \to \infty} \mathbb{E}(|X_n - X|^2) = 0$, but $\mathbb{E}[X_n^2] = \infty$ for all $n$.

**Solution.** Choose $X_n = X$ where $\mathbb{E}[X^2] = \infty$, e.g., $X$ follows Cauchy distribution.

**Problem 3**

In defining a renewal process we suppose that $F(\infty)$, the probability that an interarrival time is finite, equals 1. If $F(\infty) < 1$, then after each renewal there is a positive probability $1 - F(\infty)$ that there will be no further renewals. Argue that when $F(\infty) < 1$ the total number of renewals, call it $N(\infty)$, is such that $1 + N(\infty)$ has a geometric distribution with mean $1/(1 - F(\infty))$.

**Solution.** By definition, we see that for $k \geq 1$,
\[
\mathbb{P}(1 + N(\infty) = k) = \mathbb{P}(N(\infty) = k - 1) = F(\infty)^{k-1} (1 - F(\infty)).
\]
Therefore, $1 + N(\infty)$ follows geometric distribution with success probability $1 - F(\infty)$.

**Problem 4**

Express in words what the random variable $X_{N(t)+1}$ represents (*Hint:* It is the length of which renewal interval?) Show that
\[
P\{X_{N(t)+1} \geq x\} \geq \bar{F}(x).
\]

Compute the above exactly when $F(x) = 1 - e^{-\lambda x}$.
Solution. $X_{N(t)+1}$ is the length of the renewal interval that contains $t$. Let $F_{S_N(t)}$ denote the CDF of $S_N(t)$, then we have

$$
\mathbb{P}(X_{N(t)+1} \geq x) = \int_0^t \mathbb{P}(X_{N(t)+1} \geq x|S_N(t) = s) dF_{S_N(t)}(s)
$$

$$
= \int_0^t \mathbb{P}(X \geq x|X \geq t-s) dF_{S_N(t)}(s) = \int_0^t \frac{\bar{F}(\max(x, t-s))}{\bar{F}(t-s)} dF_{S_N(t)}(s).
$$

For any $s \in [0, t]$, it follows that $\bar{F}(\max(x, t-s))/\bar{F}(t-s) \geq \bar{F}(x)$. Hence, $\mathbb{P}(X_{N(t)+1} \geq x) \geq \bar{F}(x)$.

When $F(x) = 1 - e^{-\lambda x}$, $\bar{F}(x) = e^{-\lambda x}$. According to Ross Lemma 3.4.3,

$$
dF_{S_N(t)}(s) = \bar{F}(t)\delta(0) + \bar{F}(t-s)dm(s),
$$

where $m(s) = \mathbb{E}[N(s)] = \lambda s$. This finally leads to

$$
\mathbb{P}(X_{N(t)+1} \geq x) = \bar{F}(\max(x, t)) + \int_0^t \bar{F}(\max(x, t-s))\lambda ds
$$

$$
= \exp(-\lambda \max(x, t)) + \lambda \int_0^t \exp(-\lambda \max(x, t-s))ds
$$

$$
= \exp(-\lambda \max(x, t)) + \lambda \int_0^t \exp(-\lambda \max(x, s))ds,
$$

which can be easily computed.

**Problem 5**

Prove the renewal equation

$$
m(t) = F(t) + \int_0^t m(t-x) dF(x).
$$

**Solution.** Using the law of total expectation, we obtain that

$$
m(t) = \mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t)|X_1]] = \int_0^t \mathbb{E}[\mathbb{E}[N(t)|X_1 = x]] dF(x)
$$

$$
= \int_0^t (1 + \mathbb{E}[N(t-x)]) dF(x) = F(t) + \int_0^t m(t-x) dF(x).
$$