Problem 1 (Ross Ex. 8.6)

Let \( \{X(t), t \geq 0\} \) denote a birth and death process that is allowed to go negative and that has constant birth and death rates \( \lambda_n \equiv \lambda, \mu_n \equiv \mu, n = 0, \pm 1, \pm 2, \ldots \). Define \( \mu \) and \( c \) as functions of \( \lambda \) in such a way that \( \{cX(t), t \geq u\} \) converges to Brownian motion as \( \lambda \to \infty \). Note: The definition of birth and death process can be found in Section 5.3 of Ross.

Solution. To achieve zero mean, we should choose \( \mu = \lambda \). Then we know that the interarrival times are i.i.d. \( \text{Exp}(2\lambda) \), with mean \( 1/2\lambda \). In the time interval \([0, t]\), there are approximately \( 2\lambda t \) birth/death’s. Therefore, \( \text{Var}(X(t)) \approx 2\lambda t \) for large \( \lambda \). We should choose \( c = 1/\sqrt{2\lambda} \) such that \( \text{Var}(cX(t)) = 1 \). One can show that \( cX(t) \) weakly converges to \( B(t) \) but it is not required.

Problem 2 (Ross Ex. 8.7)

Let \( \{X(t), t \geq 0\} \) denote Brownian motion. Find the distribution of

(a) \( |X(t)| \)

(b) \( |\min_{0 \leq s \leq t} X(s)| \)

(c) \( \max_{0 \leq s \leq t} X(s) - X(t) \).

Solution. We need the following:

Lemma. (reflection principle) \( \mathbb{P}(\max_{0 \leq s \leq t} X(s) \geq a) = 2\mathbb{P}(X(t) \geq a) = \mathbb{P}(|X(t)| \geq a) \).

The random variables in (b) and (c) are equal in distribution to \( \max_{0 \leq s \leq t} X(s) \) and \( |X(t)| \).