STATS 218 Homework 8 Solutions
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Problem 1 (Grimmett Ex. 13.7.2)

Let $W$ be a standard Wiener process. Fix $t > 0$, $n \geq 1$, and let $\delta = t/n$. Show that $Z_n = \sum_{j=0}^{n-1} (W_{(j+1)\delta} - W_{j\delta})^2$ satisfies $Z_n \to t$ in mean square as $n \to \infty$.

**Solution.** Note that

$$Z_n - t = \sum_{j=0}^{n-1} \left( (W_{(j+1)\delta} - W_{j\delta})^2 - \delta \right),$$

where for each $j$, $E[(W_{(j+1)\delta} - W_{j\delta})^2] = (j+1)\delta - j\delta = \delta$, and they are mutually independent. Therefore,

$$E[(Z_n - t)^2] = \text{Var} \left( \sum_{j=0}^{n-1} (W_{(j+1)\delta} - W_{j\delta})^2 \right) = \sum_{j=0}^{n-1} \text{Var} ((W_{(j+1)\delta} - W_{j\delta})^2).$$

Since $W_{(j+1)\delta} - W_{j\delta} \sim \mathcal{N}(0, \delta)$, we know that

$$\text{Var} ((W_{(j+1)\delta} - W_{j\delta})^2) = E[(W_{(j+1)\delta} - W_{j\delta})^4] - E[(W_{(j+1)\delta} - W_{j\delta})^2]^2 = 3\delta^2 - \delta^2 = 2\delta^2,$$

since the fourth moment of $\mathcal{N}(0, 1)$ is 3. This finally leads to

$$E[(Z_n - t)^2] = 2n\delta^2 = \frac{2t^2}{n} \to 0 \text{ as } n \to \infty.$$

Problem 2 (Grimmett Ex. 13.7.3)

Let $W$ be a standard Wiener process. Fix $t > 0$, $n \geq 1$, and let $\delta = t/n$. Let $V_j = W_{j\delta}$ and $\Delta_j = V_{j+1} - V_j$.

Evaluate the limits of the following as $n \to \infty$:

(a) $I_1(n) = \sum_j V_j \Delta_j$,

(b) $I_2(n) = \sum_j V_{j+1} \Delta_j$,

(c) $I_3(n) = \sum_j \frac{1}{2} (V_{j+1} + V_j) \Delta_j$,

(d) $I_4(n) = \sum_j W_{(j+\frac{1}{2})\delta} \Delta_j$.

**Solution.** In this problem, all limits are in $L^2$ sense.

(a) By definition, $\lim_{n \to \infty} I_1(n) = \int_0^t W_s dW_s$.

(b) Note that $I_2(n) = I_1(n) + \sum_j \Delta_j^2$. According to Problem 1, $\sum_j \Delta_j^2 \to t$. Hence, $I_2(n) \to \int_0^t W_s dW_s + t$. 


(c) \( I_3(n) = (I_1(n) + I_2(n))/2 \). Therefore, \( I_3(n) \to \int_0^t W_s dW_s + t/2 \).

(d) By direct calculation, we obtain that

\[
I_4(n) - I_3(n) = \frac{1}{2} \sum_i ((W_{j+1/2})_\delta - (W_{j+1/2})_\delta) \Delta_j
\]

\[
= \frac{1}{2} \sum_j ((W_{j+1/2})_\delta - W_j)_\delta - ((W_{j+1/2})_\delta - W_{j+1/2})_\delta)
\]

\[
= \frac{1}{2} \sum_j ((W_{j+1/2})_\delta - W_j)_\delta^2 - \frac{1}{2} \sum_j ((W_{j+1/2})_\delta - W_{j+1/2})_\delta^2.
\]

Similar to the proof of Problem 1, we can show that

\[
\sum_j ((W_{j+1/2})_\delta - W_j)_\delta^2 \to \frac{t}{2}, \quad \sum_j ((W_{j+1/2})_\delta - W_{j+1/2})_\delta^2 \to \frac{t}{2},
\]

thus leading to \( I_4(n) - I_3(n) \to 0 \). Therefore, \( I_4(n) \to \int_0^t W_s dW_s + t/2 \).

**Problem 3 (Grimmett Ex. 13.12.7)**

Let \( X_0, X_1, \ldots \) be independent \( N(0, 1) \) variables, and show that

\[
W(t) = \frac{t}{\sqrt{\pi}} X_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^\infty \frac{\sin(kt)}{k} X_k
\]

defines a standard Wiener process on \([0, \pi]\).

**Solution.** First we show that the series converges uniformly (along a subsequence), implying that the limit exists and is a continuous function of \( t \). Set

\[
Z_{mn}(t) = \sum_{k=m}^{n-1} \frac{\sin(kt)}{k} X_k, \quad M_{mn} = \sup \{|Z_{mn}(t)| : 0 \leq t \leq \pi\}.
\]

We have that

\[
M_{mn}^2 \leq \sup_{0 \leq t \leq \pi} \left| \sum_{k=m}^{n-1} \frac{e^{ikt}}{k} X_k \right|^2 \leq \sum_{k=m}^{n-1} X_k^2 + 2 \sum_{l=1}^{n-1-m} \sum_{j=m}^{n-l-1} \frac{X_j X_{j+l}}{j(j+l)}.
\]

The mean value of the final term is, by the Cauchy-Schwarz inequality, no larger than

\[
2 \sum_{l=1}^{n-m-1} \sqrt{\frac{1}{j(j+l)^2}} \leq 2 \sum_{l=1}^{n-m-1} \frac{1}{j^2(j+l)^2} \leq 2(n-m) \sqrt{n-m} \sqrt{\frac{1}{m^3}}.
\]

Combine this with (*) to obtain

\[
E(M_{m,2m}^2) \leq E(M_{m,2m}^2) \leq \frac{3}{\sqrt{m}}.
\]
It follows that
\[ \mathbb{E}\left( \sum_{n=1}^{\infty} M_{2^n-1,2^n} \right) \leq \sum_{n=1}^{\infty} \frac{6}{2^n/2} < \infty, \]

implying that \( \sum_{n=1}^{\infty} M_{2^n-1,2^n} < \infty \) a.s. Therefore the series which defines \( W \) converges uniformly with probability 1 (along a subsequence), and hence \( W \) has (a.s.) continuous sample paths.

Certainly \( W \) is a Gaussian process since \( W(t) \) is the sum of normal variables (see Problem (7.11.19)). Furthermore \( \mathbb{E}(W(t)) = 0 \), and
\[
\text{cov}(W(s), W(t)) = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(ks) \sin(kt)}{k^2},
\]
since the \( X_i \) are independent with zero means and unit variances. It is an exercise in Fourier analysis to deduce that \( \text{cov}(W(s), W(t)) = \min\{s, t\} \).