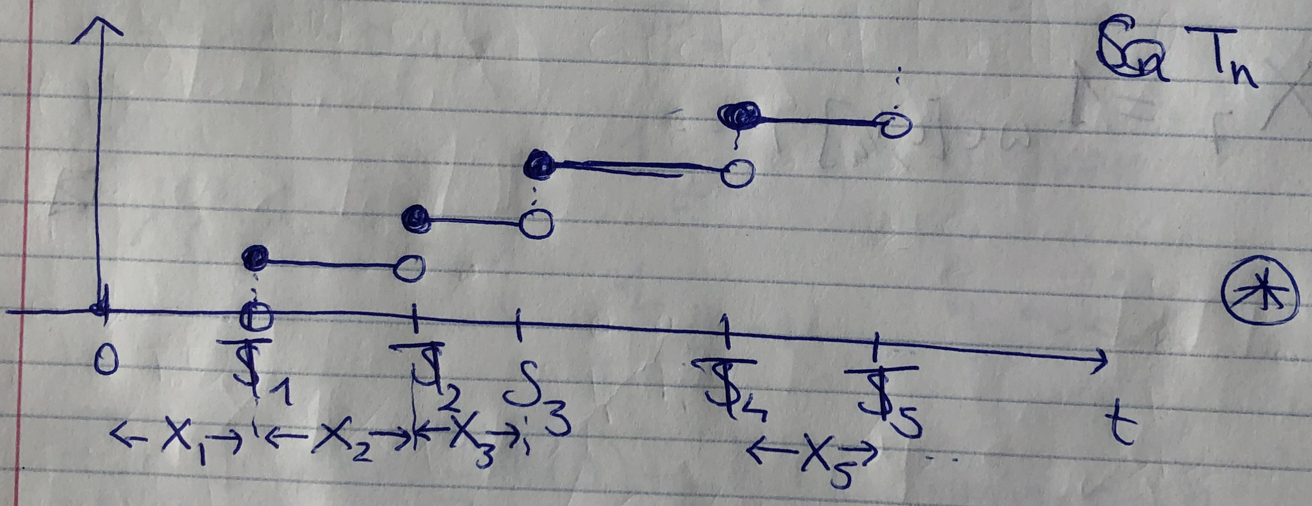
3/30/2022

Def $\{N(t)\}_{t\in[0,\infty)}$ is a renewal process if there exist iid non-negative rv's $\{X_i\}_{i\geq 1}$ st

$$N(t) = \max\{n : \underbrace{\sum_{i=1}^{n} X_i}_{T_n} \leq t\}.$$



- X_i inter-arrival times
- S_i arrival times

Notice that we can extend it to $\mathbb{R}_{<0}$ by stipulating

$\{X_i\}_{i \in \mathbb{Z}}$ iid

$S_0 = 0$

$\Phi_{-1} = X_{-1}$

$\Phi_{-k} = \sum_{\ell=+1}^{+k} X_{-\ell}$.

(in this sense (*) is the process "as seen from an arrival")

- Will assume $\mathbb{P}(X_1 = 0) < 1$ $(\Rightarrow N(t) < \infty$ a.s.$)$

<u>Example 1</u>

Continuous time Markov chain

$Z(t) \in S$ finite, $t \geq 0$

(Recall this is specified by transition rates $(q_{ij})_{ij \in S}$

$\mathbb{P}(Z(t+\delta) = j \mid \{Z(s)\}_{s \leq t}) = q_{Z(t),j} \cdot \delta + o(\delta)$

equivalently

Consider MC starting at $Z(0) = x \in S$

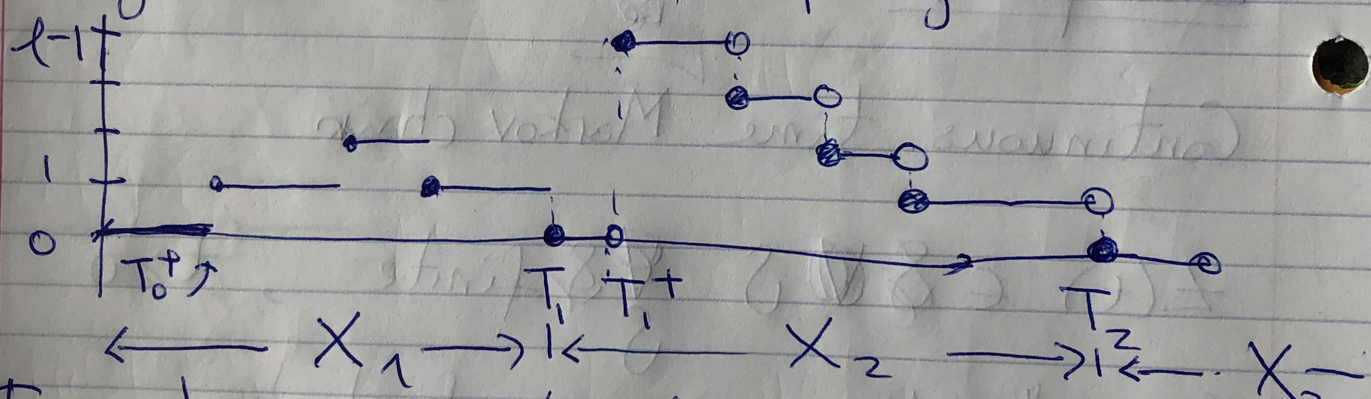
$T_0^+ = \inf\{t \geq 0 : Z(t) \neq x\}$

$T_1 = \inf\{t > T_0^+ : Z(t) = x\}$

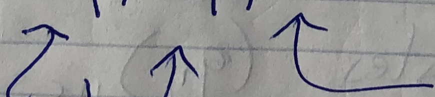
$T_i^+ := \inf\{t > T_{i-1} : Z(t) \neq x\}$

$T_{i+1} := \inf\{t > T_i^+ : Z(t) = x\}$

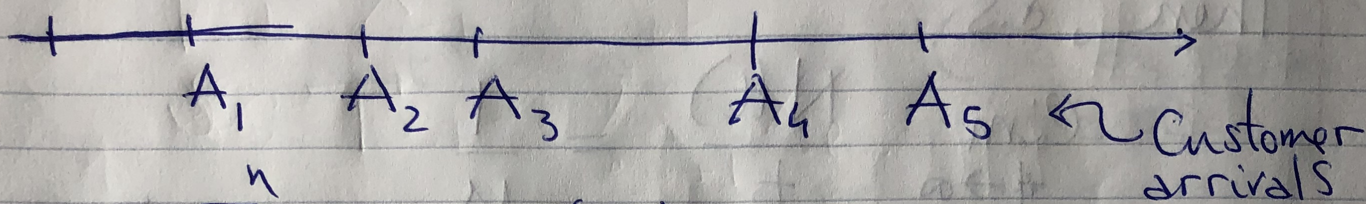
Eg rw on a circle of length l



Example 2 G/G/1 Queue

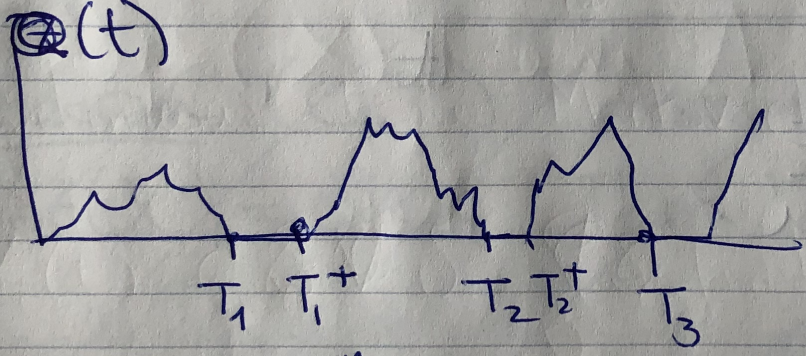


$Q(t) :=$ # of customers in the Queue at time t



$$A_n = \sum_{e=1}^{n} Y_e \qquad (Y_e)_{e \geq 1} \text{ iid}$$

$T_0 = 0,\ T_i > 0$ times at which the Queue empties



$$S_n^{(k)} := \sum_{e=1}^{n} W_e^{(k)} \qquad (W_e^{(k)} \text{ iid})$$

$$T_k^+ \leq t \leq T_{k+1}$$

$$Q(t) = \sup \{ n : A_n \leq t \}$$

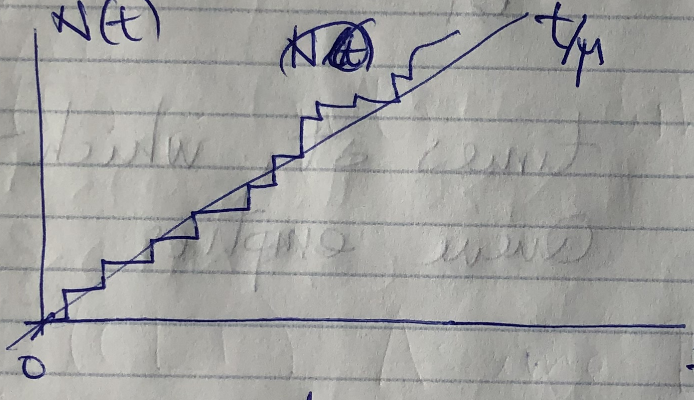
$$- \sup \{ n : A_n \leq T_k \}$$

$$- \sup \{ n : S_n^{(k)} \leq t \}$$

Theorem Assume $\mu = \mathbb{E}(X_1) < \infty$.

Then, a.s

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}.$$



$$N(t) \approx \frac{t}{\mu}$$

Proof

$$\sum_{i=1}^{N(t)} X_i \leq t \leq \sum_{i=1}^{N(t)+1} X_i$$

$$T_{N(t)} \leq t \leq T_{N(t)+1}$$

$$\frac{N(t)+1}{T_{N(t)+1}}\left(\frac{N(t)}{N(t)+1}\right) \leq \frac{N(t)}{t} \leq \frac{N(t)}{T_{N(t)}}$$

$\frac{T_n}{n} \to \mu$ a.s. (SLLN)

$N(t) \to \infty$ a.s. (To be proven) (exercise)

$\Rightarrow \frac{N(t)}{N(t)+1} \to 1, \quad \frac{N(t)}{T_{N(t)}} \to \frac{1}{\mu}.$

$\Rightarrow$ Claim

~~CLT~~

~~Theorem (CLT) Assume $\mathbb{E}X_1 = \mu < \infty$~~
~~$Var(X_1) = \sigma^2 < \infty$. Then~~

~~$$\frac{N(t) - t/\mu}{\sigma \mu^{-3/2} \sqrt{t}} \xrightarrow{d} G \sim N(0,1)$$~~

~~- Recall $\xrightarrow{d}$ means~~

~~- Informally $N(t) \approx N\left(\frac{t}{\mu}, \frac{\sigma^2 t}{\mu^{3/2}}\right)$~~