Stats 218, Introduction to Stochastic Processes II

Practice Midterm

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Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 1 hour but you are not required to solve all the problems!

Just solve those that you can solve within the time limit. Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, one of the TAs will be outside the room.

For this midterm, you are not allowed to consult textbooks or notes.

[Note: This is a practice midterm, so it is a bit longer than the actual one.]

Problem 1 (10 points)

Let G, F be two distribution functions on $\mathbb{R}_{>0}$, and $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ be independent non-negative random variables with $\mathbb{P}(X_i \leq t) = F(t)$ for all $i \geq 1$, $\mathbb{P}(Y_1 \leq t) = G(t)$, $\mathbb{P}(Y_i \leq t) = F(t)$ for all $i \geq 2$. Assume all these random variables are not identically zero.

Define the ordinary and delayed renewal processes as

$$N(t) := \max\Big\{n : \sum_{i=1}^{n} X_i \leq t\Big\}\,, \tag{1}$$

$$N_D(t) := \max\Big\{n : \sum_{i=1}^{n} Y_i \leq t\Big\}\,. \tag{2}$$

Let $m(t) := \mathbb{E}N(t)$, $m_D(t) := \mathbb{E}N_D(t)$.

Derive the equation

$$m_D(t) = G(t) + \int_{[0,t]} m(t-x)\,\mathrm{d}G(x)\,. \tag{3}$$

Problem 2 (10 points)

Let $(X_i)_{i\geq 1}$ be a sequence of independent identically distributed random variables with

$$\mathbb{P}\Big(X_i = \frac{1}{2}\Big) = \mathbb{P}\Big(X_i = \frac{3}{2}\Big) = \frac{1}{2}\,. \tag{4}$$

Define

$$Z_n := \prod_{i=1}^{n} X_i\,. \tag{5}$$

(a) Is Z_n a martingale? Justify your answer.

(b) Does $Z_n \xrightarrow{a.s.} Z_\infty$ for some random variable Z_∞? Justify your answer.

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Problem 3 (10 points)

Consider the random variables Z_n defined in last problem and, for a constant $c \in \mathbb{R}$, define

$$Y_n := \log Z_n - cn \tag{6}$$

(a) Can the constant c be chosen so that Y_n is a martingale?

(b) Let Y_n be the martingale constructed in (a). Prove that, for any $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}(Y_n > n\delta) = 0$.