These are the instructions you will see on the actual final exam. The practice final will not be graded, so you should not submit your solutions.

Instructions:

• Remember the university honor code.

• Return your solutions by email to stats300a-aut1516-staff@lists.stanford.edu or to XYZ’s office by the deadline. We strongly encourage returning solutions by email if at all possible.

• Late finals will not be accepted. If you are returning your final early, and no one is in the office, please slide your solutions under the door.

• Include your name and Stanford email handle with your solutions.

• Please write neatly: we can’t grade what we can’t read.

• To obtain full credit, you must justify your answers. We are primarily interested in your understanding of concepts, so show us what you know.

• You may refer to your textbooks and to your notes during this examination.

• You should not need to access the internet to answer any of these questions, and using the internet to search for solutions to these problems is cheating and in violation of our honor code. However, you may use Wikipedia to look up the form of unfamiliar distributions.

• Do not discuss this examination with anyone until after the deadline has elapsed.

• Unless a problem specifically forbids it, you are free to cite results proved in class, in the textbooks, or on your problem sets.
Fact: If $\gamma_i \sim \text{Gamma}(\alpha_i, 1)$ (for a shape parameter $\alpha_i$ and a scale parameter equal to 1), then $(\gamma_1, \ldots, \gamma_n) / (\sum_{j=1}^n \gamma_j)$ is distributed as a Dirichlet distribution with parameters $(\alpha_1, \ldots, \alpha_n)$, with density

$$p(v; \alpha) = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \prod_{i=1}^n v_i^{\alpha_i-1} I\left(\sum_{i=1}^n v_i = 1\right).$$

You may use this fact without proof.

Problem 1. Call an estimator $\delta(X)$ an extended Bayes estimator of $\theta$, if for any $\epsilon > 0$, there exists a proper prior $\Lambda_\epsilon$ such that

$$\int R(\theta, \delta)d\Lambda_\epsilon \leq r_{\Lambda_\epsilon} + \epsilon,$$

where $r_{\Lambda}$ represents the Bayes risk under a prior $\Lambda$. Suppose that our data $X$ is Poisson with unknown mean $\theta > 0$.

(a) (5 points) Show that $X$ is an extended Bayes estimator of $\theta$ under

$$L(\theta, d) = \frac{(\theta - d)^2}{\theta}.$$

(b) (5 points) Use part (a) to show that $X$ is minimax for $\theta$ under $L$. (Your solution must use the conclusion of part (a).)

Problem 2 (10 points). Suppose $X_1, \ldots, X_n$ are i.i.d. with gamma density given by

$$f_{b,g}(x) = \frac{1}{\Gamma(g)b^g} x^{g-1} \exp(-x/b)$$

for $x > 0$ and unknown positive parameters $b$ and $g$. For a fixed $g_0 > 0$, determine a uniformly most powerful invariant test with respect to a suitable group of transformations for testing $g \leq g_0$ versus $g > g_0$.

Problem 3. Suppose that $X_1, \ldots, X_n \sim \text{Poi}(\theta)$, and let $\phi$ be a UMP level $\alpha$ test of $\theta = 1$ versus $\theta > 1$.

(a) (5 points) Find a uniformly minimum risk mean-unbiased estimator of the power $\beta(\theta)$ of $\phi$ under the loss

$$L(\theta, d) = |\beta(\theta) - d|.$$

(b) (5 points) Is your estimator from part (a) admissible under $L$?
Problem 4. Let $N$ follow a binomial distribution with 10 trials and success probability $p \in (0, 1)$. Given $N = n$, let $X_1, \ldots, X_n$ be i.i.d. normal with mean $\theta$ and variance one. Your data consists of $(N, X_1, \ldots, X_N)$.

(a) (5 points) Show that if $p$ is known, then there does not exist a UMP level $\alpha$ test of $\theta = 0$ versus $\theta > 0$.

(b) (5 points) Supposing that $p$ is unknown, find an UMPU level $\alpha$ test of $\theta = 0$ versus $\theta > 0$.

Problem 5 (10 points). Suppose that $N$ random variables are generated i.i.d. from a distribution with known strictly increasing absolutely continuous cdf $F$, but we only observe $X$, the maximum of these random variables. Is there a UMP level $\alpha$ test of $H_0 : N \leq 5$ versus $H_1 : N > 5$? If so, find it. If not, explain why.