Stat 310A/Math 230A Theory of Probability

Practice Problems for Final

Andrea Montanari *December 3, 2019*

Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 3 hours but you are not required to solve all the problems!

Just solve those that you can solve within the time limit. Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, one of the TAs will be outside the room.

You can consult textbooks (Dembo, Billinglsely, Williams, Durrett) and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from Amir Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1 (20 points)

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the space of infinite binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$, and, for $a \le b$, write ω_a^b for the vector $(\omega_a, \omega_{a+1}, \dots, \omega_b)$. Let $\mathcal{F}$ the σ-algebra gnerated by cylindrical sets

$$C_{\ell,\xi} = \left\{\omega \in \Omega \, : \, \omega_1^\ell = \xi_1^\ell\right\}, \tag{1}$$

for $\ell \in \mathbb{N}$, $\xi \in \Omega$. Let $\mathbb{P}$ be the product measure over $(\Omega, \mathcal{F})$, defined by

$$\mathbb{P}(C_{\ell,\xi}) = \prod_{i=1}^{\ell} p(\xi_i)\,, \tag{2}$$

where $p(1) = 1 - p(0) = p \in (0,1)$. Define, for $\lambda \in (0, 1/2]$

$$X(\omega) \equiv \sum_{i=1}^{\infty} \omega_i \, \lambda^{i-1}\,, \tag{3}$$

and let $\mathcal{P}_X$ be its law.

(a) Prove that, for $\lambda = 1/2$ and any $0 < x_1 < x_2 < 2$, $\mathcal{P}_X((x_1, x_2)) > 0$. What happens if $\lambda \in (0, 1/2)$?

(b) Prove that, for $\lambda \in (0, 1/2)$, $\mathcal{P}_X$ does not have atoms. What happens if $\lambda = 1/2$?
[Recall that an atom is a Borel set $A \subseteq \mathbb{R}$ such that $\mathcal{P}_X(A) > 0$ and, for any Borel set $B \subseteq A$, $\mathcal{P}_X(B) = 0$ or $\mathcal{P}_X(B) = \mathcal{P}_X(A)$.]

Problem 2 (30 points)

Let Ω be the space of functions $\omega : [0,1] \to \mathbb{R}$, and, for each $t \in [0,1]$, define $X_t(\omega) = \omega(t)$. Let $\mathcal{F} \equiv \sigma(\{X_t\}_{t\in[0,1]})$ be the smallest σ-algebra such that X_t is measurable for each $t \in [0,1]$.

Also, for any $S \subseteq [0,1]$, le $\mathcal{F}_S \equiv \sigma(\{X_t\}_{t\in S})$ be the smallest σ-algebra such that X_t is measurable for each $t \in S$.

(a) Prove that

$$\mathcal{F} = \bigcup_{S \text{ countable}} \mathcal{F}_S\,. \tag{4}$$

(b) Show that, for any random variable Z on $(\Omega, \mathcal{F})$ there exists S countable such that Z is measurable on $(\Omega, \mathcal{F}_S)$.

(c) Define

$$Z(\omega) = \sup_{t \in [0,1]} X_t(\omega)\,. \tag{5}$$

Is Z measurable on $(\Omega, \mathcal{F})$?

Problem 3 (40 points)

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$:

$$S^{d-1} = \big\{x \in \mathbb{R}^d : ||x|| = 1\big\}\,. \tag{6}$$

The sphere S^{d-1} can be given the topology induced by $\mathbb{R}^d$. More precisely $A \subseteq S^{d-1}$ is open if for any $x \in A$, there exists $\varepsilon > 0$ such that $\{y \in S^{d-1} : ||x - y|| \le \varepsilon\} \subseteq A$.

Let $\mathcal{B}(S^{d-1})$ be the corresponding Borel σ-algebra. For any $A \in \mathcal{B}(S^{d-1})$, define

$$\Gamma(A) = \big\{rx \,:\, r \in [0,1],\, x \in A\big\}\,, \tag{7}$$

(a) Show that, for any $A \in \mathcal{B}(S^{d-1})$, $\Gamma(A) \in \mathcal{B}(\mathbb{R}^d)$.

(b) Let λ_d be the Lebesgue measure on $\mathbb{R}^d$, and define, for $A \in \mathcal{B}(S^{d-1})$,

$$\mu(A) = d\,\lambda_d(\Gamma(A))\,. \tag{8}$$

Prove that μ is a finite measure on $(S^{d-1}, \mathcal{B}(S^{d-1}))$.

(c) For $A \in \mathcal{B}(S^{d-1})$ and $0 \le a \le b$, define the set $C_{a,b}(A) \in \mathcal{B}(\mathbb{R}^d)$ as $C_{a,b}(A) = \{rx \,:\, a < r \le b\, x \in A\}$. Prove that

$$\lambda_d(C_{a,b}(A)) = \frac{b^d - a^d}{d}\,\mu(A)\,. \tag{9}$$

[Hint: Use the fact that, for $\gamma > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $\lambda_d(\gamma\, B) = \gamma^d \lambda_d(B)$ (with $\gamma\, B$ the set obtained by 'dilating' B by a factor γ).]

(d) Deduce that, for any $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\lambda_d(B) = \int_0^\infty \int_{S^{d-1}} \mathbb{I}(rx \in B)\, r^{d-1}\, \mathrm{d}\mu(x)\mathrm{d}r\,. \tag{10}$$

Problem 4 (40 points)

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \dots, \mathrm{Z}\}^{\mathbb{N}}$ the space of infinite strings of capital letters from the english alphabet (it might be useful to recall that there are 26 such letters). Further, let $\mathcal{F}$ be the σ-algebra generated by cylindrical sets (i.e. sets of the form $C_{\ell,a} = \{\omega = (\omega_1, \omega_2, \dots) : \omega_1 =$

$a_1, \dots \omega_\ell = a_\ell\}$ for some $\ell \in \mathbb{N}$ and some sequence of letters $a = (a_1, \dots, a_\ell)$), and $\mathbb{P}$ the uniform measure, defined by

$$\mathbb{P}(C_{\ell,a}) \equiv \frac{1}{26^\ell} \,. \tag{11}$$

For any $\omega \in \Omega$ and $N \in \mathbb{N}$, let $Z_N(\omega)$ be the number of occurrences of the word PROBABILITY in $(\omega_1, \dots, \omega_N)$.

(a) Show that Z_N is indeed a random variable (i.e. it is measurable on $(\Omega, \mathcal{F})$).

(b) Show that the limit $\lim_{N\to\infty} \mathbb{E}[Z_N]/N$ exists, and compute it. Call the result m.

(c) Prove that Z_N satisfies the law of large numbers, i.e. that

$$\mathbb{P}\Big\{ \lim_{N\to\infty} \frac{Z_N(\omega)}{N} = a \Big\} = 1 \,. \tag{12}$$

(d) Show that Z_N satisfies the following central limit theorem

$$\lim_{N\to\infty} \mathbb{P}\Big\{ \frac{Z_N(\omega) - Nm}{b\sqrt{N}} \le z \Big\} = F_{\rm G}(z) \,. \tag{13}$$

for some $b \in \mathbb{R}$ and all $z \in \mathbb{R}$. Here $F_{\rm G}(z) = \mathbb{P}\{Y \le z\}$ is the distribution function of a standard normal random variable Y. [Hint: Partition the string $(\omega_1 \dots \omega_N)$ into blocks.]

Problem 5 (40 points)

Let Ω be the interval $[0, 2\pi)$ with the end-points identified (in other words, this is a circle indexed by the angular coordinate). Endow this set with the standard topology, whereby a basis of neighborhoods of x is given by the intervals $(x - \varepsilon, x + \varepsilon)$ for $x \neq 0$ (and $\varepsilon > 0$ small enough) and $(2\pi - \varepsilon, \varepsilon)$ for $x = 0$. The resulting topological space Ω is compact.

The mapping $\varphi : [0, 2\pi) \to \Omega$ (the first space endowed with the standard topology), with $\varphi(x) = x$, is piecewise continuous together with its inverse. In particular both φ and φ^{-1} are measurable with respect to the Borel σ algebras. The Lebesgue measure λ_Ω is uniquely defined by $\lambda_\Omega = \lambda \circ \varphi^{-1}$. Analogously, for any measure ν on $([0, 2\pi), \mathcal{B}_{[0,2\pi)})$ one can associate the measure $\nu \circ \varphi^{-1}$ on $(\Omega, \mathcal{B}_\Omega)$.

Given a probability measure μ on $(\Omega, \mathcal{B}_\Omega)$, its Fourier coefficients are the numbers

$$c_k(\mu) = \int_\Omega e^{ikx}\, \mu({\rm d}x) \,, \tag{14}$$

for $k \in \mathbb{Z}$. It is known that, for any $0 < a < b < 2\pi$ with $\mu(\{a\}) = \mu(\{b\}) = 0$,

$$\mu((a, b]) = \lim_{m\to\infty} \int_{(a,b]} \Big\{ \frac{1}{2m\pi} \sum_{l=0}^{m-1} \sum_{k=-l}^{l} c_k\, e^{-ikt} \Big\} {\rm d}t \,, \tag{15}$$

where $c_k = c_k(\mu)$. (You are welcome to use this fact in answering the following questions.)

(a) Show that the Fourier coefficients uniquely determine the probability measure, i.e. that given μ, ν probability measures on $(\Omega, \mathcal{B}_\Omega)$ with $c_k(\mu) = c_k(\nu)$ for all $k \in \mathbb{Z}$, we have $\mu = \nu$.

(b) Given two independent random variables X, Y taking values in Ω, let $Z = X \oplus Y$ be defined by

$$X \oplus Y = \begin{cases} X + Y & \text{if } X + Y \in [0, 2\pi), \\ X + Y - 2\pi & \text{if } X + Y \in [2\pi, 4\pi). \end{cases} \tag{16}$$

Can you express the Fourier coefficients of (the law of) Z in terms of (the laws of) X and Y.

(c) Let $\{X_i\}_{i\in\mathbb{N}}$, be independent and identically distributed random variables taking values in Ω, and assume their common distribution to admit a density f_X with respect to the Lebesgue measure. Let $\mu^{(n)}$ be the law of $X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

Prove that, as $n \to \infty$, $\mu^{(n)}$ converges weakly to the uniform distribution over Ω (i.e. to $U = \lambda_\Omega/(2\pi)$).

(d) Consider now the case in which $X_i = \theta$ for all i almost surely, for some $\theta \in [0, 2\pi)$ with θ/π *irrational*. Does $\mu^{(n)}$ have a weak limit? Consider the average

$$\nu^{(n)} \equiv \frac{1}{n} \sum_{k=1}^{n} \mu^{(k)} \,. \tag{17}$$

Does $\nu^{(n)}$ have a weak limit as $n \to \infty$? Prove your answer.