Stat 310A/Math 230A Theory of Probability

Practice Final Solutions

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Problem 1

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the space of infinite binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$, and, for $a \le b$, write ω_a^b for the vector $(\omega_a, \omega_{a+1}, \dots, \omega_b)$. Let $\mathcal{F}$ the σ-algebra gnerated by cylindrical sets

$$C_{\ell,\xi} = \left\{\omega \in \Omega \,:\, \omega_1^\ell = \xi_1^\ell\right\}, \tag{1}$$

for $\ell \in \mathbb{N}$, $\xi \in \Omega$. Let $\mathbb{P}$ be the product measure over $(\Omega, \mathcal{F})$, defined by

$$\mathbb{P}(C_{\ell,\xi}) = \prod_{i=1}^{\ell} p(\xi_i)\,, \tag{2}$$

where $p(1) = 1 - p(0) = p \in (0,1)$. Define, for $\lambda \in (0, 1/2]$

$$X(\omega) \equiv \sum_{i=1}^{\infty} \omega_i\, \lambda^{i-1}\,, \tag{3}$$

and let $\mathcal{P}_X$ be its law.

(a) Prove that, for $\lambda = 1/2$ and any $0 < x_1 < x_2 < 2$, $\mathcal{P}_X((x_1, x_2)) > 0$. What happens if $\lambda \in (0, 1/2)$?

Solution : Assume, without loss of generality $|x_2 - x_1| \ge 2^{-n+1}$. Then there exists an integer $k \in \{1, \dots, 2^n - 1\}$, such that $x_1 < k \cdot 2^{-n} < (k+1)2^{-n} < x_2$. Of course

$$\mathcal{P}_X((x_1, x_2)) \ge \mathbb{P}(k \cdot 2^{-n} \le X(\omega) \le (k+1)2^{-n})\,. \tag{4}$$

The integer k admits the unique binary expansion $k = \sum_{i=1}^{n} k_i 2^{n-i}$. Then

$$\mathbb{P}(k \cdot 2^{-n} \le X(\omega) \le (k+1)2^{-n}) = \mathbb{P}(C_{n,(k_1 \dots, k_n)}) = p^{n_1(k)}(1-p)^{n_0(k)}\,, \tag{5}$$

with $n_0(k)$ and $n_1(k)$ the number of zeros and ones in $(k_1, \dots, k_n)$. For $p \in (0,1)$ the above probability is strictly positive.

For $\lambda \in (0, 1/2)$, we have

$$X(\omega) \le \sum_{i=1}^{\infty} \lambda^{i-1} = \frac{1}{1-\lambda} < 2\,. \tag{6}$$

Hence we have $\mathcal{P}_X((x_1, x_2)) = 0$ if $x_1 > (1-\lambda)^{-1}$.

(b) Prove that, for $\lambda \in (0, 1/2)$, $\mathcal{P}_X$ does not have atoms. What happens if $\lambda = 1/2$?
[Recall that an atom is a Borel set $A \subseteq \mathbb{R}$ such that $\mathcal{P}_X(A) > 0$ and, for any Borel set $B \subseteq A$, $\mathcal{P}_X(B) = 0$ or $\mathcal{P}_X(B) = \mathcal{P}_X(A)$.]

Solution : For $n \geq 1$, define

$$X_n(\omega) \equiv \sum_{i=1}^{n-1} \omega_i \, \lambda^i \,. \tag{7}$$

Obviously $X_n(\omega) \leq X(\omega) \leq X_n(\omega) + (1-\lambda)^{-1}\lambda^n$, whence, for any interval $[a, b) \subseteq \mathbb{R}$

$$\mathbb{P}\{X(\omega) \in [a, b)\} \leq \mathbb{P}\{X_n(\omega) \in [a - \delta_n, b)\} \,, \tag{8}$$

with $\delta_n \equiv (1-\lambda)^{-1}\lambda^n \leq 2\lambda^n$. In particular,

$$\mathbb{P}\{X(\omega) \in [a, a + \lambda^n)\} \leq \mathbb{P}\{X_n(\omega) \in [a - 2\lambda^n, a + \lambda^n)\} \,. \tag{9}$$

For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, let $J_n(a) \equiv [a, a + C\lambda^n)$, with $C = (1-2\lambda)/(1-\lambda) > 0$. If for any $\omega \in \Omega$, $X(\omega) \notin J_n(a)$, then $\mathbb{P}\{X(\omega) \in J_n(a)\} = 0$. Assume therefore, that there is at least one ralization $\omega^* = (\omega_1^*, \dots, \omega_n^*, \dots)$ such that $X_n(\omega^*) \in J_n(\lambda)$. For any $\omega \neq \omega_*$, let $k = k(\omega)$ be the smallest index such that $\omega_k^* \neq \omega_k$. Then

$$|X(\omega) - X(\omega^*)| \geq \lambda^k - \sum_{l=k+1}^{\infty} \lambda^l = C(\lambda)\, \lambda^k \,. \tag{10}$$

Therefore $X(\omega) \in J_n(a)$ only if the first n coordinates of ω coincide with those of ω^*, i.e.

$$\mathbb{P}\{X(\omega) \in J_n(a)\} \leq \mathbb{P}\{\omega_1 = \omega_1^*, \dots, \omega_n = \omega_n^*\} \leq \max(p, 1-p)^n \,. \tag{11}$$

As a consequence, for any $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that $\mathbb{P}\{X_n(\omega) \in [a, a + \delta(\varepsilon))\} \leq \varepsilon$.

This immediately implies that $\mathcal{P}_X$ does not have atoms. Indeed, assume this is not the case and let S be such an atom, with $\mathcal{P}_X(S) = 2\varepsilon$. Obviously $S \subseteq [0, 2]$. Partition the interval $[0, 2]$ into intervals $J_1, J_2, \dots, J_M$ of length $\delta(\varepsilon)$. Then $\mathcal{P}_X(J_i \cap S) > 0$ for at least one interval i. On the other hand $\mathcal{P}_X(J_i \cap S) \leq \mathcal{P}_X(J_i) \leq \varepsilon$.

For the case $\lambda = 1/2$, the claim follows by proving that X is uniformly random in the interval $[0, 2)$. This in turn follows by checking that $\mathbb{P}(X \in [i/2^n, (i+1)/2^n)) = 1/2^{n+1}$, for all $n \geq 1$, and $i \in \{0, \dots, 2^{n+1} - 1\}$.

Problem 2

Let Ω be the space of functions $\omega : [0, 1] \to \mathbb{R}$, and, for each $t \in [0, 1]$, define $X_t(\omega) = \omega(t)$. Let $\mathcal{F} \equiv \sigma(\{X_t\}_{t \in [0,1]})$ be the smallest σ-algebra such that X_t is measurable for each $t \in [0, 1]$.

Also, for any $S \subseteq [0, 1]$, le $\mathcal{F}_S \equiv \sigma(\{X_t\}_{t \in S})$ be the smallest σ-algebra such that X_t is measurable for each $t \in S$.

(a) Prove that

$$\mathcal{F} = \bigcup_{S \text{ countable}} \mathcal{F}_S \,. \tag{12}$$

Solution: Let $\mathcal{A} \equiv \bigcup_{S \text{ countable}} \mathcal{F}_S$. It is clear that X_t is measurable on $\mathcal{A}$ for each $t \in [0, 1]$. Indeed, $\mathcal{A}$ contains in particular $\mathcal{F}_{\{t\}} = \sigma(X_t)$.

Further $\mathcal{A} \subseteq \mathcal{F}$, since $\mathcal{F}_S \subseteq \mathcal{F}$ for each $S \subseteq [0, 1]$ (indeed $\mathcal{F}_S$ is the *minimal* σ algebra such that X_t is measurable for each $t \in S$).

The claim follows if we show that $\mathcal{A}$ is a σ-algebra. Let $B \in \mathcal{A}$. Then $B \in \mathcal{F}_S$ for some S countable, whence $B^{\rm c} \in \mathcal{F}_S$ (because $\mathcal{F}_S$ is a σ-algebra) and thus $B^{\rm c} \in \mathcal{A}$. Therefore $\mathcal{A}$ is closed under complements.

Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable collection in $\mathcal{A}$. Then there exist countable sets $S_i \subseteq [0,1]$ such that $B_i \in \mathcal{F}_{S_i}$ for each i. In particular $B_i \in \mathcal{F}_S$ with $S = \cup_{i=1}^{\infty} S_i$. Let $B \equiv \cup_{i=1}^{\infty} B_i$. By the σ-algebra property, $B \in \mathcal{F}_S$ as well. But S is countable (countable union of countable sets), whence $B \in \mathcal{A}$.

(b) Show that, for any random variable Z on $(\Omega, \mathcal{F})$ there exists S countable such that Z is measurable on $(\Omega, \mathcal{F}_S)$.

Solution: Let $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ be an ordering of the rationals. By point (a) above, for each i, there exist S_i countable, such that the set $B_i = \{\omega : Z(\omega) \le q_i\}$ is in in $\mathcal{F}_{S_i}$. As a consequence for each i, $B_i \in \mathcal{F}_S$ with $S \equiv \cup_{i=1}^{\infty} S_i$. This imply that $\{Z^{-1}((-\infty, q]) : q \in \mathbb{Q}\} \subseteq \mathcal{F}_S$. Since $\mathcal{P} = \{(-\infty, q] : q \in \mathbb{Q}\}$ is a π system which generates the Borel σ-algebra, the thesis follows.

(c) Define

$$Z(\omega) = \sup_{t\in[0,1]} X_t(\omega)\,. \tag{13}$$

Is Z measurable on $(\Omega, \mathcal{F})$?

Solution: No, it is not measurable. Indeed, assume by contradiction that it is measrable. Then by point (b) above, there exist S countable such that Z is measurable on $\mathcal{F}_S$. Consider the set $B = \{\omega : Z(\omega) \le 0\}$, and let ω_1, ω_2 be two functions such that $\omega_1(t) = \omega_2(t) \le 0$ for all $t \in S$ and $\sup_{t\in[0,1]} \omega_1(t) > 0 \ge \sup_{t\in[0,1]} \omega_2(t)$. Then of course $\omega_1 \notin B$, $\omega_2(t)$. On the other hand, for any $A \in \mathcal{F}_S$ either $\omega_1, \omega_2 \in S$ or $\omega_1, \omega_2 \not{i}nS$, which leads to a contradiction. (The last claim follows from Problem 2 in the midterm.)

Problem 3

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$:

$$S^{d-1} = \big\{x \in \mathbb{R}^d : ||x|| = 1\big\}\,. \tag{14}$$

The sphere S^{d-1} can be given the topology induced by $\mathbb{R}^d$. More precisely $A \subseteq S^{d-1}$ is open if for any $x \in A$, there exists $\varepsilon > 0$ such that $\{y \in S^{d-1} : ||x-y|| \le \varepsilon\} \subseteq A$.

Let $\mathcal{B}(S^{d-1})$ be the corresponding Borel σ-algebra. For any $A \in \mathcal{B}(S^{d-1})$, define

$$\Gamma(A) = \big\{rx : r \in [0,1],\, x \in A\big\}\,, \tag{15}$$

(a) Show that, for any $A \in \mathcal{B}(S^{d-1})$, $\Gamma(A) \in \mathcal{B}(\mathbb{R}^d)$.

Solution: For $\varepsilon > 0$, let $\Gamma_\varepsilon(A) \equiv \{rx : r \in (\varepsilon, 1],\, x \in A\}$. Then $\Gamma_\varepsilon(A) = f_\varepsilon^{-1}(A)$, for the continuous mapping $f_\varepsilon : \{x \in \mathbb{R}^d : \varepsilon \le ||x|| \le 1\} \to S^{d-1}$, $x \mapsto x/||x||$. Since counterimages of Borel sets under continuous mappings are Borel, we have $\Gamma_\varepsilon(A) \in \mathcal{B}(\mathbb{R}^d)$. The thesis follows since

$$\Gamma(A) = \bigcup_{n=1}^{\infty} \Gamma_{1/n}(A) \cup \{0\}\,. \tag{16}$$

(b) Let λ_d be the Lebesgue measure on $\mathbb{R}^d$, and define, for $A \in \mathcal{B}(S^{d-1})$,

$$\mu(A) = d\, \lambda_d(\Gamma(A))\,. \tag{17}$$

Prove that μ is a finite measure on $(S^{d-1}, \mathcal{B}(S^{d-1}))$.

Solution: Obviously μ is a non-negative set function, with $\mu(\emptyset) = d\lambda_d(\emptyset) = 0$. If $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}(S^{d-1})$ is a disjoint collection than $\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ are also disjoint with $B_i = \Gamma(A_i)\setminus\{0\}$. Further $\Gamma(\cup_i A_i) = \cup_i \Gamma(A_i)$. Therefore, since $\lambda_d(\{0\}) = 0$, we have

$$\mu(\cup_{i\geq 1} A_i) = d\lambda_d(\cup_{i\geq 1}\Gamma(A_i)) = d\lambda_d(\cup_{i\geq 1} B_i) = \sum_{i\geq 1} d\lambda_d(B_i) = \sum_{i\geq 1} d\lambda_d(\Gamma(A_i)) = \sum_{i\geq 1} \mu(A_i)\,, \tag{18}$$

i.e. μ is countably additive, hence a measure.

Finally $\mu(S^{d-1}) = d\lambda_d(\{x : ||x|| \leq 1\}) \leq d\lambda_d(\{x : \max_i |x_i| \leq 1\}) = d2^d$. Therefore μ is finite.

(c) For $A \in \mathcal{B}(S^{d-1})$ and $0 \leq a \leq b$, define the set $C_{a,b}(A) \in \mathcal{B}(\mathbb{R}^d)$ as $C_{a,b}(A) = \{rx \,:\, a < r \leq b\, x \in A\}$. Prove that

$$\lambda_d(C_{a,b}(A)) = \frac{b^d - a^d}{d}\,\mu(A)\,. \tag{19}$$

[Hint: Use the fact that, for $\gamma > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $\lambda_d(\gamma\, B) = \gamma^d \lambda_d(B)$ (with $\gamma\, B$ the set obtained by 'dilating' B by a factor γ).]

Solution: First consider the case $b = 1$, $a/b = \alpha < 1$. Using the definition of $\Gamma_\varepsilon(A)$ in point (a), we have $\Gamma_0(A) = \cup_{i=0}^{\infty} C_{\alpha^{i+1},\alpha^i}(A)$. Since the union is disjoint, and $\lambda_d(\{0\}) = 0$, we have

$$\mu(A) = d\lambda_d(\Gamma_0(A)) = \sum_{i=0}^{\infty} d\,\lambda_d(C_{\alpha^{i+1},\alpha^i}(A)) = \sum_{i=0}^{\infty} d\,\alpha^{id}\lambda_d(C_{\alpha,1}(A)) = \frac{1}{1-\alpha^d}\, d\lambda_d(C_{\alpha,1}(A))\,. \tag{20}$$

For $b \neq 1$, it is sufficient to use $\lambda_d(C_{a,b}(A)) = b^d \lambda_d(C_{\alpha,1}(A))$ for $\alpha = a/b$.

(d) Deduce that, for any $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\lambda_d(B) = \int_0^\infty \int_{S^{d-1}} \mathbb{I}(rx \in B)\, r^{d-1}\, \mathrm{d}\mu(x)\mathrm{d}r\,. \tag{21}$$

Solution: We can assume $0 \notin B$, since both sides are modified by a vanishing term . Let $\omega(B)$ be the quantity defined on the right hand side of Eq. (21). Notice, by Fubini, that $\omega(B)$ is the integral of the simple function $\mathbb{I}(rx \in B)$ under the product measure $\mu \times \lambda_1$ on $S^{d-1} \times (0, \infty)$. Therefore ω is a measure on $\mathcal{B}(\mathbb{R}^d)$. Further, both λ_d and ω are σ-finite (it is sufficient to consider the sets $B_n \equiv \{x : ||x|| \leq n\} \uparrow \mathbb{R}^d$. Finally, by point (c) above

$$\lambda_d(C_{a,b}(A)) = \omega(C_{a,b}(A))\,, \tag{22}$$

for any $a < b$, $A \in \mathcal{B}(S^{d-1})$. The thesis follows by showing that $\mathcal{P} = \{C_{a,b}(A) \,:\, a < b,\ A \in \mathcal{B}(S^{d-1})\}$ is a π-system (this is obvious) that generates $\mathcal{B}(\mathbb{R}^d)$.

There are many ways of proving the last claim. One is the following. First define, for $A \in \mathcal{B}(S^{d-1})$,

$$D_{a,b}(A) = \{rx \,:\, a < r < b\, x \in A\}\,. \tag{23}$$

It is clear that $D_{a,b}(A)$ can be constructed by finite intersections and unions of sets $\{C_{a,b}(A)\}$. Consider next any open set $Q \subseteq \mathbb{R}^d$. We want to show that it is a countable union of sets $\{D_{a,b}(A)\}$ with A relatively open in S^{d-1}. Without loss of generality we can assume $0 \notin Q$ and $Q \subseteq H_\varepsilon$ with $H_\varepsilon \equiv \{x \in \mathbb{R}^d : x_1 \geq \varepsilon\}$ an half space. Let $\psi : H_\varepsilon \to \mathbb{R}^d$ be the mapping

$$\psi(x_1, \dots, x_d) = (r(x), x_2/r(x), \dots, x_d/r(x)), \tag{24}$$

$$r(x) \equiv \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}, \tag{25}$$

which is differentiable together with its inverse on $\psi(H_\varepsilon)$. The set $\psi(Q)$ is open in $\mathbb{R}^d$. Therefore

$$\psi(Q) = \bigcup_{i=1}^{\infty} R_i, \tag{26}$$

with the R_i's open rectangles in $\mathbb{R}^d$ (because rectangles generate the Borel σ-algebra). Therefore

$$Q = \bigcup_{i=1}^{\infty} \psi^{-1}(R_i), \tag{27}$$

but $\psi^{-1}(R_i) = D_{a_i,b_i}(A_i)$ for some a_i, b_i, A_i.

Problem 4

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \dots, \mathrm{Z}\}^{\mathbb{N}}$ the space of infinite strings of capital letters from the english alphabet (it might be useful to recall that there are 26 such letters). Further, let $\mathcal{F}$ be the σ-algebra generated by cylindrical sets (i.e. sets of the form $C_{\ell,a} = \{\omega = (\omega_1, \omega_2, \dots) : \omega_1 = a_1, \dots \omega_\ell = a_\ell\}$ for some $\ell \in \mathbb{N}$ and some sequence of letters $a = (a_1, \dots, a_\ell)$), and $\mathbb{P}$ the uniform measure, defined by

$$\mathbb{P}(C_{\ell,a}) \equiv \frac{1}{26^\ell}. \tag{28}$$

For any $\omega \in \Omega$ and $N \in \mathbb{N}$, let $Z_N(\omega)$ be the number of occurrences of the word PROBABILITY in $(\omega_1, \dots, \omega_N)$.

(a) Show that Z_N is indeed a random variable (i.e. it is measurable on $(\Omega, \mathcal{F})$).

Solution: Let $X_n(\omega)$ be the indicator on the event

$$\{\omega_{n-10} = \mathrm{P}, \omega_{n-9} = \mathrm{R}, \omega_{n-8} = \mathrm{O}, \omega_{n-7} = \mathrm{B}, \omega_{n-6} = \mathrm{A}, \omega_{n-5} = \mathrm{B}, \omega_{n-4} = \mathrm{I}, \omega_{n-3} = \mathrm{L}, \omega_{n-2} = \mathrm{I}, \omega_{n-1} = \mathrm{T}, \omega_n = \mathrm{Y}\},$$

with, by convention $X_n(\omega) = 0$ for $n \leq 10$. Clearly, X_n is an indicator on finite union of cylinder sets, hence it is measurable. Further

$$Z_N(\omega) = \sum_{n=1}^{N} X_n(\omega), \tag{29}$$

whence Z_N is also measurable.

(b) Show that the limit $\lim_{N\to\infty} \mathbb{E}[Z_N]/N$ exists, and compute it. Call the result m.

Solution : By independence, we have, for any $n > 10$, $\mathbb{E}[X_n] = 1/26^{11}$. Therefore $\mathbb{E}[Z_N] = (N-10)/26^{11}$, which immediately implies the thesis with $a = 1/26^{11}$.

(c) Prove that Z_N satisfies the law of large numbers, i.e. that

$$\mathbb{P}\Big\{ \lim_{N\to\infty} \frac{Z_N(\omega)}{N} = a \Big\} = 1 \,. \tag{30}$$

Solution : Let $Y_n \equiv X_n - a$. Then,

$$\mathbb{E}\Big\{ \Big(\frac{Z_N}{N} - a \Big)^4 \Big\} = \frac{1}{N^4} \sum_{i,j,k,l=11}^{N} \mathbb{E}\{Y_i Y_j Y_k Y_l\} \le \frac{24}{N^4} \sum_{11\le i\le j\le k\le l\le N}^{N} |\mathbb{E}\{Y_i Y_j Y_k Y_l\}| \,. \tag{31}$$

Notice that $\mathbb{E}(Y_i) = 0$, $|Y_i| \le 1$ and Y_i is independendent from Y_j, Y_k, Y_l unless $j - i \le 10$. Analogously Y_l is independendent from Y_i, Y_j, Y_k unless $l - k \le 10$. Therefore

$$\mathbb{E}\Big\{ \Big(\frac{Z_N}{N} - a \Big)^4 \Big\} \le \frac{24}{N^4} \sum_{11\le i\le j\le k\le l\le N} \mathbb{I}(j-i\le 10)\mathbb{I}(l-k\le 10) \le \frac{24\cdot 11^2}{N^4} \sum_{1\le j\le k\le N} 1 \le \frac{2000}{N^2} \,. \tag{32}$$

By Markov inequality for any $\varepsilon > 0$, $\mathbb{P}\{|Z_N/N - a| \ge \varepsilon\} \le C(\varepsilon)/N^2$. Applying Borel-Cantelli I we obtain the desired result.

(d) Show that Z_N satisfies the following central limit theorem

$$\lim_{N\to\infty} \mathbb{P}\Big\{ \frac{Z_N(\omega) - Nm}{b\sqrt{N}} \le z \Big\} = F_{\rm G}(z) \,. \tag{33}$$

for some $b \in \mathbb{R}$ and all $z \in \mathbb{R}$. Here $F_{\rm G}(z) = \mathbb{P}\{Y \le z\}$ is the distribution function of a standard normal random variable Y. [Hint: Partition the string $(\omega_1 \dots \omega_N)$ into blocks.]

Solution : Throughout we let $S_N = Z_N(\omega) - Na = \sum_{n=11}^{N} Y_n$. We want to prove that

$$\lim_{N\to\infty} \mathbb{P}\big\{ S_N(\omega)/b\sqrt{N} \le z \big\} = F_{\rm G}(z) \,. \tag{34}$$

Fix $\gamma \in (0, 1/2)$ and let $m \equiv \lfloor N^{1/2-\gamma} \rfloor$. Partition the set $\{11, \dots, N\}$ into m consecutive intervals, each of length $\ell \equiv \lfloor (N-10)/m \rfloor$ or $\ell + 1$, to be denoted by $J_1, J_2, \dots, J_m$ (that is $J_1 = \{11, \dots, 11 + \ell - 1\}$), etc). Partition each of these intervals into two consecutive intervals as $J_i = K_i \cup L_i$ with $|L_i| = 10$ or 11 and $|K_i| = \ell - 10$. Define

$$W_i = \sum_{n\in K_i} Y_n \,, \qquad S_N^* = \sum_{i=1}^{m} W_i \,. \tag{35}$$

The W_i's are independent and identically distributed with $\mathbb{E}W_i = 0$. Further, proceeding as in point (b) above, it is easy to see that

$$\mathbb{E}(W_i^2) \;\equiv\; b_\ell \ell = b\ell + O(1) \,, \tag{36}$$

$$\mathbb{E}(W_i^4) \;\le\; c\ell^2 \,. \tag{37}$$

Consider therefore the normalized sum $\widehat{S}_N^* = \sum_{i=1}^{m} W_i/\sqrt{Nb}$. The Lindeberg parameter reads

$$g_N(\varepsilon) = \frac{1}{Nb} \sum_{i=1}^{m} \mathbb{E}\big\{ W_i^2 : |W_i| \ge \varepsilon\sqrt{Nb} \big\} \le \frac{1}{(N\varepsilon b)^2} \sum_{i=1}^{m} \mathbb{E}\big\{ W_i^4 \big\} \le \frac{cm\ell^2}{(N\varepsilon b)^2} \le \frac{c'}{\varepsilon^2 m} \,, \tag{38}$$

Since $m \to \infty$ as $N \to \infty$, we have $g_N(\varepsilon) \to 0$. Further $\mathrm{Var}(\widehat{S}_N^*) = m\mathbb{E}(W_i^2)/(Nb) \to 1$ because $\ell \to \infty$ as well. By Lindeberg central limit theorem

$$\lim_{N\to\infty} \mathbb{P}\big\{S_N^*/b\sqrt{N} \le z\big\} = F_{\rm G}(z)\,. \tag{39}$$

Since $|Y_n| \le 1$, we have $|S_N - S_N^*| \le 11\,m \le \delta\,\sqrt{N}$, for any $\delta > 0$ and all $N > N_0(\delta)$. Therefore

$$\mathbb{P}\big\{S_N^* \le zb\sqrt{N} - \delta\sqrt{N}\big\} \le \mathbb{P}\big\{S_N \le zb\sqrt{N}\big\} \le \mathbb{P}\big\{S_N^* \le zb\sqrt{N} + \delta\sqrt{N}\big\}\,. \tag{40}$$

By taking the limit $N \to \infty$ and using Eq. (39), we get

$$F_{\rm G}(z-\delta) \le \lim\inf_{N\to\infty} \mathbb{P}\big\{S_N(\omega)/b\sqrt{N} \le z\big\} \le \lim\sup_{N\to\infty} \mathbb{P}\big\{S_N(\omega)/b\sqrt{N} \le z\big\} \le F_{\rm G}(z+\delta)\,. \tag{41}$$

The thesis follows by taking $\delta \to 0$ by continuity of $F_{\rm G}$.

Problem 5

Let Ω be the interval $[0, 2\pi)$ with the end-points identified (in other words, this is a circle indexed by the angular coordinate). Endow this set with the standard topology, whereby a basis of neighborhoods of x is given by the intervals $(x-\varepsilon, x+\varepsilon)$ for $x \neq 0$ (and $\varepsilon > 0$ small enough) and $(2\pi - \varepsilon, \varepsilon)$ for $x = 0$. The resulting topological space Ω is compact.

The mapping $\varphi : [0, 2\pi) \to \Omega$ (the first space endowed with the standard topology), with $\varphi(x) = x$, is piecewise continuous together with its inverse. In particular both φ and φ^{-1} are measurable with respect to the Borel σ algebras. The Lebesgue measure λ_Ω is uniquely defined by $\lambda_\Omega = \lambda \circ \varphi^{-1}$. Analogously, for any measure ν on $([0, 2\pi), \mathcal{B}_{[0,2\pi)})$ one can associate the measure $\nu \circ \varphi^{-1}$ on $(\Omega, \mathcal{B}_\Omega)$.

Given a probability measure μ on $(\Omega, \mathcal{B}_\Omega)$, its Fourier coefficients are the numbers

$$c_k(\mu) = \int_\Omega e^{ikx}\,\mu(\mathrm{d}x)\,, \tag{42}$$

for $k \in \mathbb{Z}$. It is known that, for any $0 < a < b < 2\pi$ with $\mu(\{a\}) = \mu(\{b\}) = 0$,

$$\mu((a,b]) = \lim_{m\to\infty} \int_{(a,b]} \Big\{\frac{1}{2m\pi} \sum_{l=0}^{m-1} \sum_{k=-l}^{l} c_k\, e^{-ikt}\Big\}\mathrm{d}t\,, \tag{43}$$

where $c_k = c_k(\mu)$. (You are welcome to use this fact in answering the following questions.)

(a) Show that the Fourier coefficients uniquely determine the probability measure, i.e. that given μ, ν probability measures on $(\Omega, \mathcal{B}_\Omega)$ with $c_k(\mu) = c_k(\nu)$ for all $k \in \mathbb{Z}$, we have $\mu = \nu$.

Solution: For $z \in [0, 2\pi)$, let $G(z) = \mu([0, z))$. It is clearly sufficient to show that G is uniquely determined by the Fourier coefficients, since the intervals $[0, z)$ form a π-system that generates $\mathcal{B}_\Omega$. By assumption $G(0) = 0$ and $G(2\pi) = 1$. Further G is non-dereasing and right-continuous. Let $\mathcal{C}$ be the set of continuity points $a \in (0, 2\pi)$ such that $\mu(\{a\}) = 0$. For $a \in \mathcal{C}$, $1 - G(a) = \mu((a, 2\pi))$ is uniquely determined by the inversion formula (43) as the limit for $b \uparrow 2\pi$, $b \in \mathcal{C}$ of $\mu((a, b])$. For general a, using right continuity we have $G(a) = \inf\{G(a') : a' > a, a' \in \mathcal{C}\}$.

Therefore G is uniquely detemined by the Fourier coefficients.

(b) Given two independent random variables X, Y taking values in Ω, let $Z = X \oplus Y$ be defined by

$$X \oplus Y = \begin{cases} X + Y & \text{if } X + Y \in [0, 2\pi), \\ X + Y - 2\pi & \text{if } X + Y \in [2\pi, 4\pi). \end{cases} \tag{44}$$

Can you express the Fourier coefficients of (the law of) Z in terms of (the laws of) X and Y.

Solution: We have, for $k \in \mathbb{Z}$, $c_k(Z) = \mathbb{E}\{e^{ikZ}\}$. But $Z = X + Y - 2\pi\ell$ for an integer ℓ, and therefore $e^{ikZ} = e^{ik(X+Y)}$. Using independence $c_k(Z) = \mathbb{E}\{e^{ikZ}\} = \mathbb{E}\{e^{ikX}e^{ikY}\} = \mathbb{E}\{e^{ikX}\}\mathbb{E}\{e^{ikY}\} = c_k(X)c_k(Y)$.

(c) Let $\{X_i\}_{i\in\mathbb{N}}$, be independent and identically distributed random variables taking values in Ω, and assume their common distribution to admit a density f_X with respect to the Lebesgue measure. Let $\mu^{(n)}$ be the law of $X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

Prove that, as $n \to \infty$, $\mu^{(n)}$ converges weakly to the uniform distribution over Ω (i.e. to $U = \lambda_\Omega/(2\pi)$).

Solution: Let $c_k^{(n)} = c_k(\mu^{(n)})$. We claim that, for any $k \in \mathbb{Z}$, $c_k^{(n)} \to c_k(U)$. Since Ω is compact, the sequence of probability measures $\mu^{(n)}$ is uniformly tight. Hence any subsequence $\{\mu^{(n(m))}\}$ admits a converging subsequence $\mu^{(n'(m))} \overset{w}{\Rightarrow} \nu$, with $\{n'(m)\}_{m\in\mathbb{N}} \subseteq \{n(m)\}_{m\in\mathbb{N}}$. Since $x \mapsto e^{ikx}$ is a continuous bounded function, $c_k^{n'(m)} \to c_k(\nu)$ along such a subsequence. But as proved in point (a), the Fourier coefficiends determine uniquely the distribution, whence $\nu = U$ for any subsequence. Therefore (by the same argument as in Levy's continuity theorem) $\mu^{(n)} \overset{w}{\Rightarrow} U$

We are left with the task of proving $c_k^{(n)} \to c_k(U)$. Notice that $c_0(U) = 1$ and $c_k(U) = 0$ for $k \neq 0$. Let $c_k^{(n)} = \int e^{ikx}\mu^{(n)}(\mathrm{d}x)$. By point (b) above $c_k^{(n)} = (c_k)^n$ for $c_k = \mathbb{E}\{e^{ikX}\}$. Clearly $c_0 = 1$. It is therefore sufficient to prove that $|c_k| < 1$ for all $k \neq 0$. Using Fubini, we get immediately $|c_k|^2 = \mathbb{E}\{e^{ik(X-Y)}\} = \mathbb{E}\{\cos k(X - Y)\}$ for X, Y i.i.d. with density f_X. Therefore, since $(\cos(\alpha/2))^2 = (1 - \cos\alpha)/2$ and using the fact that X, Y have a density

$$1 - |c_k|^2 = \int_{[0,2\pi)\times[0,2\pi)} \Big(\cos\frac{k(x-y)}{2}\Big)^2 f(x)\, f(y)\, \mathrm{d}x \times \mathrm{d}y \tag{45}$$

for $\mathrm{d}x \times \mathrm{d}y$ the Lebesgue measure in $\mathbb{R}^2$. Therefore $|c_k| = 1$ implies $f(x)f(y) = 0$ for almost every (x, y), i.e. $f(x) = 0$ for almost every x, which is impossible since $\int f(x) = 1$. This implies $|c_k| < 1$ as claimed.

(d) Consider now the case in which $X_i = \theta$ for all i almost surely, for some $\theta \in [0, 2\pi)$ with θ/π *irrational*. Does $\mu^{(n)}$ have a weak limit? Consider the average

$$\nu^{(n)} \equiv \frac{1}{n}\sum_{k=1}^{n} \mu^{(k)}\,. \tag{46}$$

Does $\nu^{(n)}$ have a weak limit as $n \to \infty$? Prove your answer.

Solution: We have $\mu^{(n)} = \delta_{x_n}$ for $x_n = n\theta - \ell 2\pi$ (with an appropriate choice of ℓ). For θ/π irrational the sequence x_n does not converge, and hence $\mu^{(n)}$ does not converge either.

We claim that $\nu^{(n)}$ converges weakly to U (the uniform probability measure over $[0, 2\pi)$). By the same argument as in point (c) above, it is sufficient to prove that the corresponding Fourier coefficients $c_k^{(n)} = \int e^{ikx}\nu^{(n)}(\mathrm{d}x)$ are such that $c_k^{(n)} \to 0$ for all $k \neq 0$ (obviously $c_0^{(n)} = 1$).

We have $\nu^{(n)} = n^{-1}\sum_{\ell=1}^{n}\delta_{x_\ell}$. For k integer $e^{ikx_\ell} = e^{ik\ell\theta}$. Therefore, for $k \neq 0$,

$$c_k^{(n)} = \frac{1}{n}\sum_{\ell=1}^{n} e^{2\pi ik\ell\theta} = \frac{1}{n}\,\frac{e^{ik\theta} - e^{ik(n+1)\theta}}{1 - e^{ik\theta}}\,, \tag{47}$$

whence $|c_k^{(n)}| \le 2/(n(1 - \cos k\theta)) \to 0$ (because for θ/π irrational, $\cos(k\theta) < 1$ for all k).
