Stat 310A/Math 230A Theory of Probability

Homework 1

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- **(Important)** Return all assignments with a cover page stapled to the front of the work that details only basic information: name, quarter, course.
- Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) for each problem.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- In some cases, multiple homework options will be proposed (and indicated as 'Option 1', 'Option 2', etc.). You are welcome to work on all the problems proposed (solutions will be posted), but should submit only those corresponding to one 'Option'.

Option 1: Exercises on measure spaces

Solve Exercises [1.1.4], [1.1.13], [1.1.21], [1.1.22] and [1.1.33] in Amir Dembo's lecture notes.

Option 2: The Banach-Tarski paradox in one dimension

The objective of this homework is to prove a simplified version of the Banach-Tarski paradox, in the case of the real line. The definition of equidecomposable subsets of $\mathbb{R}$ is provided below.

Definition 1. *The sets $A, B \in \mathbb{R}$ are (countably) equidecomposable if there exist countable partitions*

$$A = \cup_{i=1}^{\infty} A_i\,, \qquad B = \cup_{i=1}^{\infty} B_i\,, \tag{1}$$

and real numbers $\{t_1, t_2, t_3 \dots\}$ such that for every $i \in \mathbb{N}$ $A_i = B_i + t_i$ ($+t_i$ here indicates translation by t_i).

It might also be useful to recall the Axiom of Choice.

Axiom 2. *Let Ω be a set and $\mathcal{C} = \{A_\alpha\}_{\alpha\in\Gamma}$ be a collection of* nonempty *subsets $A_\alpha \subseteq \Omega$. Then there exists at least one* choice function*, i.e. a function $f : \mathcal{C} \to \Omega$ such that*

$$f(A) \in A\,, \tag{2}$$

for each $A \in \mathcal{C}$.

First, we start with some useful remarks. Here A, B, A_i, B_i are subsets of $\mathbb{R}$. Further, for $t \in \mathbb{R}$, $R_t : \mathbb{R} \to \mathbb{R}$ is the translation by t: $R_t(x) = x + t$.

A1 We will say that a function $f : A \to \mathbb{R}$ is a (countable) *equidecomposition* if there exists a countable partition $A = \cup_{i=1}^{\infty} A_i$, and reals $\{t_i\}_{i\in\mathbb{N}}$ such that $f|_{A_i} = R_{t_i}|_{A_i}$ for each i.

Show that A is equidecomposable with B if and only if there exists an equidecomposition $f : A \to B$ which is bijective.

A2 Let $A' \subseteq A$ and $B' \subseteq B$, and assume there exist bijective equidecompositions $f : A \to B'$ and $g : B \to A'$.

Construct an equidecomposition $h : A \to B$, and prove that it is bijective.

[Hint: Let $A^{(0)} \equiv A \setminus g(B)$, and $A^{(*)} \equiv \cup_{n=0}^{\infty}(g \circ f)^n(A^{(0)})$. Define $h(x) = f(x)$ if $x \in A^{(*)}$ and $h(x) = g^{-1}(x)$ if $x \in A \setminus A^{(*)}$.]

Next to the actual problem:

B1 Use the axiom of choice to show that there exists $C \subseteq [0, 1/2]$ such that the following is a partition

$$\mathbb{R} = \cup_{x \in C}\{x + \mathbb{Q}\} \,. \tag{3}$$

B2 Show that $\mathbb{Q} \cap [0, 1/2]$ is equidecomposable with $\mathbb{Q}$.

B3 Deduce that

$$A \equiv \cup_{x \in C}\{x + (\mathbb{Q} \cap [0, 1/2])\} \subseteq [0, 1] \tag{4}$$

is equidecomposable with $\mathbb{R}$.

B4 Use A1, A2 above to show that this implies that $[0, 1]$ is equidecomposable with $\mathbb{R}$. What does this result imply for measures on $\mathbb{R}$?