Option 1: Exercises on measure spaces

Exercise [1.1.4]

1. A and $B \setminus A$ are disjoint with $B = A \cup (B \setminus A)$ so $P(A) + P(B \setminus A) = P(B)$ and rearranging gives the desired result.

2. Let $A_n' = A_n \cap A$, $B_1 = A'_1$ and for $n > 1$, $B_n = A_n' \setminus \cup_{m=1}^{n-1} A_m'$. Since the $B_n$ are disjoint and have union $A$ we have using (a) and $B_m \subseteq A_m$

$$P(A) = \sum_{m=1}^{\infty} P(B_m) \leq \sum_{m=1}^{\infty} P(A_m)$$

3. Consider the disjoint sets $B_n = A_n \setminus A_{n-1}$ for which $\cup_{m=1}^{\infty} B_m = A$, and $\cup_{m=1}^{n} B_m = A_n$. Then,

$$P(A) = \sum_{m=1}^{\infty} P(B_m) = \lim_{n \to \infty} \sum_{m=1}^{n} P(B_m) = \lim_{n \to \infty} P(A_n)$$

4. $A_n' \uparrow A'$, so (c) implies $P(A_n') \uparrow P(A')$. Since $P(B') = 1 - P(B)$ it follows that $P(A_n) \downarrow P(A)$.

Exercise [1.1.13]

(a) Let $G = \bigcap_{\alpha} \mathcal{F}_{\alpha}$, with each $\mathcal{F}_{\alpha}$ a $\sigma$-algebra. Since $\mathcal{F}_{\alpha}$ a $\sigma$-algebra, we have that $\Omega \in \mathcal{F}_{\alpha}$, and as this applies for all $\alpha$, it follows that $\Omega \in G$. Suppose now that $A \in G$. That is, $A \in \mathcal{F}_{\alpha}$ for all $\alpha$. Since each $\mathcal{F}_{\alpha}$ is a $\sigma$-algebra, it follows that $A^c \in \mathcal{F}_{\alpha}$ for all $\alpha$, and hence $A^c \in G$. Similarly, let $A = \bigcup_{i} A_i$ for some countable collection $A_1, A_2, \ldots$ of elements of $G$. By definition of $G$, necessarily $A_i \in \mathcal{F}_{\alpha}$ for all $i$ and every $\alpha$. Since $\mathcal{F}_{\alpha}$ is a $\sigma$-algebra, we deduce that $A \in \mathcal{F}_{\alpha}$, and as this applies for all $\alpha$, it follows that $A \in G$.

(b) We verify the conditions for $\sigma$-algebra.

(a) $\Omega \in G$ and $\Omega \cap H = H \in \mathcal{H}$. Hence $\Omega \in \mathcal{H}^H$.

(b) Suppose $A \in \mathcal{H}^H$. Since $G$ is a $\sigma$-algebra and $A \in G$, we have $A^c \in G$. Note that $A^c \cap H = (A \cap H)^c \cap H$. Since by definition $A \cap H \in H$, we have $A^c \cap H \in \mathcal{H}$ as well. Hence $A^c \in \mathcal{H}^H$.

(c) Suppose $A_i \in \mathcal{H}^H$ for $i \in \mathbb{N}$. Since $A_i \in G$, $\bigcup_{i} A_i \in G$. Also, $(\bigcup_{i} A_i) \cap H = \bigcup_{i} (A_i \cap H) \in \mathcal{H}$ since each component $A_i \cap H \in \mathcal{H}$. Thus, $\bigcup_{i} A_i \in \mathcal{H}^H$.

Therefore, $\mathcal{H}^H$ as defined is a $\sigma$-algebra.

(c) Suppose we have $H_1 \subseteq H_2$. We want to show that $\mathcal{H}^{H_2} \subseteq \mathcal{H}^{H_1}$. In fact, given any $A \in \mathcal{H}^{H_2}$, since $H_1 \subseteq H_2$, we have $A \cap H_1 = (A \cap H_2) \cap H_1$, $A \cap H_2 \in \mathcal{H}$ by definition and we also know $H_1 \in \mathcal{H}$. This implies $A \cap H_1 \in \mathcal{H}$. Also, $A \in G$ by definition. Thus, $A \in \mathcal{H}^{H_1}$. Since the choice of $A$ is arbitrary, we
conclude $H^2 \subseteq H^1$.

$H^\Omega = \{A \in G : A \cap \Omega \in H\} = \{A \in G : A \in H\} = H$. On the other hand, $H^\emptyset = \{A \in G : A \cap \emptyset \in H\} = \{A \in G : \emptyset \in H\} = \emptyset$ due to the fact that whichever $A$ is chosen in $G$, $\emptyset$ is always in $H$.

First note $H \subseteq H \cup H'$. By the monotonicity derived above, $H^{H \cup H'} \subseteq H^H$. For the same reason, $H^{H \cup H'} \subseteq H^{H \cup H'}$. This results in one direction, $H^{H \cup H'} \subseteq H^H \cup H^{H'}$. We are left to prove the other direction. In fact, if $A \in H^H \cap H^{H'}$, we have $A \cap H \in H$ and $A \cap H' \in H$, and thus $A \cap (H \cup H') = (A \cap H) \cup (A \cap H') \in H$. By definition, we know $A \in H^{H \cup H'}$. Therefore, $H^H \cap H^{H'} \subseteq H^{H \cup H'}$. We conclude $H^H \cap H^{H'} = H^{H \cup H'}$.

Exercise [1.1.21]

It suffices to show that if $F$ is the $\sigma$-algebra generated by $\{(a_1, b_1) \times \cdots \times (a_d, b_d)\}$, then $F$ contains (a) the open sets and (b) all sets of the form $A_1 \times \cdots \times A_d$ where $A_i \in B$. For (a), note that if $G$ is open and $x \in G$ then there is a set of the form $\{(a_1, b_1) \times \cdots \times (a_d, b_d)\}$ with $a_i, b_i \in \mathbb{Q}$ that contains $x$ and lies in $G$, so any open set is a countable union of these basic sets $\{(a_1, b_1) \times \cdots \times (a_d, b_d)\}$ with $a_i, b_i \in \mathbb{Q}$. In this argument we relied on the fact that there are only countably many such basic sets, hence we are not bothered by the fact that there are uncountably many points $x$ in $G$.

For (b), fix $A_1, \ldots, A_d$ and let $G = \{A : A \times A_2 \times \cdots \times A_d \in F\}$. Since $F$ is a $\sigma$-algebra it is easy to see that if $\mathbb{R} \in G$ then $G$ is a $\sigma$-algebra so if $G \supseteq A$ then $G \supseteq \sigma(A)$. Applying this for $A_i = (a_i, b_i)$, $i = 2, \ldots, d$ it follows that if $A_1 \in B$ then $A_1 \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in F$. Repeating now the preceding argument for $G = \{A : A_1 \times A_2 \cdots \times A_d \in F\}$, $A_1 \in B$ and $A_i = (a_i, b_i)$, shows that if $A_1, A_2 \in B$, then $A_1 \times A_2 \times (a_3, b_3) \times \cdots \times (a_d, b_d) \in F$. Applying this type of argument $d - 2$ more times, proves the assertion (b).

Exercise [1.1.22]

We have $F = \sigma(A_\alpha, \alpha \in \Gamma)$, and want to show that every set $B$ in $F$ has a certain property. The property in this problem is $B \in \sigma(\{A_\alpha, j \geq 1\})$, for some countable $\{\alpha_j\} \subset \Gamma$, but ignore that for now, because the method indicated here applies very generally, and will be used again. Notice first that every set $A_\alpha$ in the generating class has the property. Now consider the class $C$ of all sets in $F$ that have the property. We have already shown that each $A_\alpha$ is in this class; the problem is to show that all sets in $F$ are in this class. Luckily, the “property” is such that $C$ is a $\sigma$-algebra (check: this is the only calculation in this problem). So $C$ is a $\sigma$-algebra which contains all the $A_\alpha$, hence it contains $F$, because $F$ is the intersection of all $\sigma$-algebras that contain all the $A_\alpha$.

Exercise [1.1.33]

Let $\Omega = \{1, 2, 3, 4\}$ and $A = \{\{1, 3\}, \{2, 3\}, \Omega\}$ for which $\sigma(A) = 2^\Omega$. Define $\mu$ and $\nu$ by $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 1/4$ and $\nu(\{1\}) = \nu(\{2\}) = 1/3$, $\nu(\{3\}) = \nu(\{4\}) = 1/6$.

Option 2: The Banach-Tarski paradox in one dimension

A1

For the first direction, let $f : A \rightarrow \mathbb{R}$ be an equidecomposition. and define $B_i = f(A_i) = B_i + t_i$ for $i \in \mathbb{N}$. Since $\{A_i\}$ is a countable partition of $A$, it is sufficient to show that $\{B_i\}$ is a countable partition of $B$. Indeed $B_i \cap B_j = \emptyset$ for $i \neq j$ follows from the injectivity of $f$ (because otherwise there would be $y \in B_i \cap B_j$ whence $y = f(x_i)$, and $y = f(x_j)$ for some $x_i \in A_i, x_j \in A_j$ distinct). Further, $f(A) = \bigcup_i f(A_i) = \bigcup_i B_i$, and since $f$ is surjective, $f(A) = B$. 
To prove the converse, assume \( \{A_i\} \) and \( \{B_i\} \) to be partitions (respectively) of \( A \) and \( B \), and let \( \{t_i\} \) be the such that \( B_i = A_i + t_i \). Define \( f \) by letting \( f|_{A_i} = R_{t_i}|_{A_i} \). This map is clearly bijective (with \( f^{-1}_{B_i} = R_{-t_i}|_{A_i} \)).

A2

Let \( A' \subseteq A \), \( B' \subseteq B \), and consider the bijective equidecompositions \( f : A \to B' \) and \( g : B \to A' \).

As suggested, we define \( A^{(0)} = A \setminus g(B) \), and \( A^{(*)} = \bigcup_{n=0}^{\infty} (g \circ f)^n(A^{(0)}) \). Let \( h : A \to B \) be defined by

\[
h(x) = \begin{cases} f(x) & \text{if } x \in A^{(*)} \\ g^{-1}(x) & \text{if } x \in A \setminus A^{(*)}.\end{cases}
\]

Notice that \( h \) is well defined because \( A \setminus A^{(*)} \subseteq A \setminus A^{(0)} \). Further, it is a countable equidecomposition. To prove this, consider the partitions \( A = \bigcup_{i=1}^{\infty} A_i \) and \( B = \bigcup_{i=1}^{\infty} B_i \), with respect to which \( f \) and \( g \) are (respectively) equidecompositions with translation parameters \( \{t_i\} \) and \( \{s_i\} \). Then

\[
A = \left\{ \bigcup_{i=1}^{\infty} (A_i \cap A^{(*)}) \right\} \bigcup \left\{ \bigcup_{i=1}^{\infty} (g(B_i) \cap (A \setminus A^{(*)})) \right\}
\]

is a countable partition of \( A \) and it is easy to check that \( h \) is an equidecomposition with respect to this partition. Indeed \( h|_{A_i \cap A^{(*)}} = f|_{A_i \cap A^{(*)}} = R_{t_i}|_{A_i \cap A^{(*)}} \) and \( h|_{g(B_i) \cap (A \setminus A^{(*)})} = g^{-1}|_{g(B_i) \cap (A \setminus A^{(*)})} = R_{-s_i}|_{g(B_i) \cap (A \setminus A^{(*)})} \).

It remains to prove that \( h \) is bijective. To this end, define the mapping \( l : B \to A \) by

\[
l(y) = \begin{cases} f^{-1}(y) & \text{if } g(y) \in A^{(*)} \\ g(y) & \text{otherwise.}\end{cases}
\]

It is not hard to prove that \( l \) is the inverse of \( h \). Indeed, if \( x \equiv g(y) \notin A^{(*)} \), then \( h(x) = g^{-1}(x) = y \). On the other hand, if \( g(y) \in A^{(*)} \), then \( g(y) = (g \circ f)^k(A^{(0)}) \) for some \( k \geq 1 \) (because \( A^{(0)} \cap g(B) = \emptyset \)). By injectivity of \( g \), \( y = f((g \circ f)^{k-1}(x_0)) \) for some \( x_0 \in A^{(0)} \). If we let \( x = l(y) = f^{-1}(y) \), then \( x \in A^{(*)} \) as well, whence \( h(x) = f(x) = y \).

B1

Consider the equivalence relation \( x \sim y \) if \( (x - y) \in \mathbb{Q} \), and let \( \mathcal{E} \) denote the collection of its equivalence classes. For every \( E \in \mathcal{E} \), \( E \cap [0, 1/2) \) is non-empty and by the axiom of choice, there exist a choice function \( E \mapsto x_E \) such that \( x_E \in E \cap [0, 1/2) \) for each \( E \). Obviously \( E = x_E + \mathbb{Q} \). Therefore

\[
\mathbb{R} = \bigcup_{E \in \mathcal{E}} E = \bigcup_{E \in \mathcal{E}} \{x_E + \mathbb{Q}\} = \bigcup_{x \in \mathbb{C}} \{x + \mathbb{Q}\},
\]

where \( C = \{x_E\}_{E \in \mathcal{E}} \).

B2

Since the rationals are countable, there exist partitions in isolated points

\[
\mathbb{Q} \cap [0, 1/2) = \bigcup_{i=1}^{\infty} \{q_i\}, \quad \mathbb{Q} = \bigcup_{i=1}^{\infty} \{p_i\}.
\]

Of course, for any \( i \), \( \{p_i\} = R_{t_i}(\{q_i\}) \) if we set \( t_i = p_i - q_i \).

B3

Using the enumerations of \( \mathbb{Q} \cap [0, 1/2] \) and of \( \mathbb{Q} \) at the previous point, we get

\[
A = \bigcup_{x \in \mathbb{C}} \{x + (\mathbb{Q} \cap [0, 1/2])\} = \bigcup_{i=1}^{\infty} \{q_i + C\}, \quad \mathbb{R} = \bigcup_{x \in \mathbb{C}} \{x + \mathbb{Q}\} = \bigcup_{i=1}^{\infty} \{p_i + C\}.
\]

We have \( \{p_i + C\} = R_{t_i}(\{q_i + C\}) \) if we set \( t_i = p_i - q_i \) as above.
B4

Clearly $[0, 1]$ is equidecomposable with $[0, 1] \subseteq \mathbb{R}$ (via the identity mapping). On the other hand $\mathbb{R}$ is equidecomposable with $A \subseteq [0, 1]$. By points A1, A2, this implies that $[0, 1]$ is equidecomposable with $\mathbb{R}$.

This implies that there exists no measure on $\mathbb{R}$ satisfying the following requirements

1. The measure is countably additive (As it should be).
2. $\mu([0, 1]) \notin \{0, \infty\}$.
3. Any set is measurable.
4. $\mu(S) = \mu(R_tS)$ for any measurable set $S$. 