Stat 310A/Math 230A Theory of Probability

Homework 1 Solutions

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Option 1: Exercises on measure spaces

Exercise [1.1.4]

1. A and $B \setminus A$ are disjoint with $B = A \cup (B \setminus A)$ so $\mathbf{P}(A) + \mathbf{P}(B \setminus A) = \mathbf{P}(B)$ and rearranging gives the desired result.

2. Let $A'_n = A_n \cap A$, $B_1 = A'_1$ and for $n > 1$, $B_n = A'_n \setminus \cup_{m=1}^{n-1} A'_m$. Since the B_n are disjoint and have union A we have using (a) and $B_m \subseteq A_m$

$$\mathbf{P}(A) = \sum_{m=1}^{\infty} \mathbf{P}(B_m) \leq \sum_{m=1}^{\infty} \mathbf{P}(A_m)$$

3. Consider the disjoint sets $B_n = A_n \setminus A_{n-1}$ for which $\cup_{m=1}^{\infty} B_m = A$, and $\cup_{m=1}^{n} B_m = A_n$. Then,

$$\mathbf{P}(A) = \sum_{m=1}^{\infty} \mathbf{P}(B_m) = \lim_{n\to\infty} \sum_{m=1}^{n} \mathbf{P}(B_m) = \lim_{n\to\infty} \mathbf{P}(A_n)$$

4. $A_n^c \uparrow A^c$, so (c) implies $\mathbf{P}(A_n^c) \uparrow \mathbf{P}(A^c)$. Since $\mathbf{P}(B^c) = 1 - \mathbf{P}(B)$ it follows that $\mathbf{P}(A_n) \downarrow \mathbf{P}(A)$.

Exercise [1.1.13]

(a) Let $\mathcal{G} = \bigcap_\alpha \mathcal{F}_\alpha$, with each $\mathcal{F}_\alpha$ a σ-algebra. Since $\mathcal{F}_\alpha$ a σ-algebra, we have that $\Omega \in \mathcal{F}_\alpha$, and as this applies for all α, it follows that $\Omega \in \mathcal{G}$. Suppose now that $A \in \mathcal{G}$. That is, $A \in \mathcal{F}_\alpha$ for all α. Since each $\mathcal{F}_\alpha$ is a σ-algebra, it follows that $A^c \in \mathcal{F}_\alpha$ for all α, and hence $A^c \in \mathcal{G}$. Similarly, let $A = \bigcup_i A_i$ for some countable collection $A_1, A_2, \ldots$ of elements of $\mathcal{G}$. By definition of $\mathcal{G}$, necessarily $A_i \in \mathcal{F}_\alpha$ for all i and every α. Since $\mathcal{F}_\alpha$ is a σ-algebra, we deduce that $A \in \mathcal{F}_\alpha$, and as this applies for all α, it follows that $A \in \mathcal{G}$.

(b) We verify the conditions for σ-algebra.

  (a) $\Omega \in \mathcal{G}$ and $\Omega \cap H = H \in \mathcal{H}$. Hence $\Omega \in \mathcal{H}^H$.

  (b) Suppose $A \in \mathcal{H}^H$. Since $\mathcal{G}$ is a σ-algebra and $A \in \mathcal{G}$, we have $A^c \in \mathcal{G}$. Note that $A^c \cap H = (A \cap H)^c \cap H$. Since by definition $A \cap H \in \mathcal{H}$, we have $A^c \cap H \in \mathcal{H}$ as well. Hence $A^c \in \mathcal{H}^H$.

  (c) Suppose $A_i \in \mathcal{H}^H$ for $i \in \mathbb{N}$. Since $A_i \in \mathcal{G}$, $\bigcup_i A_i \in \mathcal{G}$. Also, $(\bigcup_i A_i) \cap H = \bigcup_i (A_i \cap H) \in \mathcal{H}$ since each component $A_i \cap H \in \mathcal{H}$. Thus, $\bigcup_i A_i \in \mathcal{H}^H$.

Therefore, $\mathcal{H}^H$ as defined is a σ-algebra.

(c) Suppose we have $H_1 \subseteq H_2$. We want to show that $\mathcal{H}^{H_2} \subseteq \mathcal{H}^{H_1}$. In fact, given any $A \in \mathcal{H}^{H_2}$, since $H_1 \subseteq H_2$, we have $A \cap H_1 = (A \cap H_2) \cap H_1$. $A \cap H_2 \in \mathcal{H}$ by definition and we also know $H_1 \in \mathcal{H}$. This implies $A \cap H_1 \in \mathcal{H}$. Also, $A \in \mathcal{G}$ by definition. Thus, $A \in \mathcal{H}^{H_1}$. Since the choice of A is arbitrary, we conclude $\mathcal{H}^{H_2} \subseteq \mathcal{H}^{H_1}$.

$\mathcal{H}^{\Omega} = \{A \in \mathcal{G} : A \cap \Omega \in \mathcal{H}\} = \{A \in \mathcal{G} : A \in \mathcal{H}\} = \mathcal{H}$. On the other hand, $\mathcal{H}^{\emptyset} = \{A \in \mathcal{G} : A \cap \emptyset \in \mathcal{H}\} = \{A \in \mathcal{G} : \emptyset \in \mathcal{H}\} = \mathcal{G}$ due to the fact that whichever A is chosen in $\mathcal{G}$, $\emptyset$ is always in $\mathcal{H}$.

First note $H \subseteq H \cup H'$. By the monotonicity derived above, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^{H}$. For the same reason, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^{H'}$. This results in one direction, $\mathcal{H}^{H \cup H'} \subseteq \mathcal{H}^{H} \cap \mathcal{H}^{H'}$. We are left to prove the other direction. In fact, if $A \in \mathcal{H}^{H} \cap \mathcal{H}^{H'}$, we have $A \cap H \in \mathcal{H}$ and $A \cap H' \in \mathcal{H}$, and thus $A \cap (H \cup H') = (A \cap H) \cup (A \cap H') \in \mathcal{H}$. By definition, we know $A \in \mathcal{H}^{H \cup H'}$. Therefore, $\mathcal{H}^{H} \cap \mathcal{H}^{H'} \subseteq \mathcal{H}^{H \cup H'}$. We conclude $\mathcal{H}^{H} \cap \mathcal{H}^{H'} = \mathcal{H}^{H \cup H'}$.

Exercise [1.1.21]

It suffices to show that if $\mathcal{F}$ is the σ-algebra generated by $\{(a_1, b_1) \times \cdots \times (a_d, b_d)\}$, then $\mathcal{F}$ contains (a) the open sets and (b) all sets of the form $A_1 \times \cdots \times A_d$ where $A_i \in \mathcal{B}$. For (a), note that if G is open and $x \in G$ then there is a set of the form $(a_1, b_1) \times \cdots \times (a_d, b_d)$ with $a_i, b_i \in \mathbb{Q}$ that contains x and lies in G, so any open set is a countable union of these basic sets ($(a_1, b_1) \times \cdots \times (a_d, b_d)$ with $a_i, b_i \in \mathbb{Q}$). In this argument we relied on the fact that there are only countably many such basic sets, hence we are not bothered by the fact that there are uncountably many points x in G.

For (b), fix $A_2, \cdots, A_d$ and let $\mathcal{G} = \{A : A \times A_2 \times \cdots \times A_d \in \mathcal{F}\}$. Since $\mathcal{F}$ is a σ-algebra it is easy to see that if $\mathbb{R} \in \mathcal{G}$ then $\mathcal{G}$ is a σ-algebra so if $\mathcal{G} \supseteq \mathcal{A}$ then $\mathcal{G} \supseteq \sigma(\mathcal{A})$. Applying this for $A_i = (a_i, b_i)$, $i = 2, \ldots, d$ it follows that if $A_1 \in \mathcal{B}$ then $A_1 \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \mathcal{F}$. Repeating now the preceding argument for $\mathcal{G} = \{A : A_1 \times A \times A_3 \cdots \times A_d \in \mathcal{F}\}$, $A_1 \in \mathcal{B}$ and $A_i = (a_i, b_i)$, $i = 3, \ldots, d$, shows that if $A_1, A_2 \in \mathcal{B}$, then $A_1 \times A_2 \times (a_3, b_3) \times \cdots \times (a_d, b_d) \in \mathcal{F}$. Applying this type of argument $d-2$ more times, proves the assertion (b).

Exercise [1.1.22]

We have $\mathcal{F} = \sigma(A_\alpha, \alpha \in \Gamma)$, and want to show that every set B in $\mathcal{F}$ has a certain property. The property in this problem is $B \in \sigma(\{A_{\alpha_j}, j \geq 1\})$, for some countable $\{\alpha_j\} \subset \Gamma$, but ignore that for now, because the method indicated here applies very generally, and will be used again. Notice first that every set A_α in the generating class has the property. Now consider the class $\mathcal{C}$ of all sets in $\mathcal{F}$ that have the property. We have already shown that each A_α is in this class; the problem is to show that all sets in $\mathcal{F}$ are in this class. Luckily, the "property" is such that $\mathcal{C}$ is a σ-algebra (check: this is the only calculation in this problem). So $\mathcal{C}$ is a σ-algebra which contains all the A_α, hence it contains $\mathcal{F}$, because $\mathcal{F}$ is the intersection of all σ-algebras that contain all the A_α.

Exercise [1.1.33]

Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{A} = \{\{1, 3\}, \{2, 3\}, \Omega\}$ for which $\sigma(\mathcal{A}) = 2^\Omega$. Define μ and ν by $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = 1/4$ and $\nu(\{1\}) = \nu(\{2\}) = 1/3$, $\nu(\{3\}) = \nu(\{4\}) = 1/6$.

Option 2: The Banach-Tarski paradox in one dimension

A1

For the first direction, let $f : A \to \mathbb{R}$ be an equidecomposition. and define $B_i = f(A_i) = B_i + t_i$ for $i \in \mathbb{N}$. Since $\{A_i\}$ is a countable partition of A, it is sufficient to show that $\{B_i\}$ is a countable partition of B. Indeed $B_i \cap B_j = \emptyset$ for $i \neq j$ follows from the injectivity of f (because otherwise there would be $y \in B_i \cap B_j$ whence $y = f(x_i)$, and $y = f(x_j)$ for some $x_i \in A_i$, $x_j \in A_j$ distict). Further, $f(A) = \bigcup_i f(A_i) = \bigcup_i B_i$, and since f is surjective, $f(A) = B$.

To prove the converse, assume $\{A_i\}$ and $\{B_i\}$ to be partitions (respectively) of A and B, and let $\{t_i\}$ be the such that $B_i = A_i + t_i$. Define f by letting $f|_{A_i} = R_{t_i}|_{A_i}$. This map is clearly bijective (with $f^{-1}|_{B_i} = R_{-t_i}|_{A_i}$).

A2

Let $A' \subseteq A$, $B' \subseteq B$, and consider the bijective equidecompositions $f : A \to B'$ and $g : B \to A'$.

As suggested, we define $A^{(0)} \equiv A \setminus g(B)$, and $A^{(*)} \equiv \cup_{n=0}^{\infty}(g \circ f)^n(A^{(0)})$. Let $h : A \to B$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A^{(*)}, \\ g^{-1}(x) & \text{if } x \in A \setminus A^{(*)}. \end{cases} \tag{1}$$

Notice that h is well defined because $A \setminus A^{(*)} \subseteq A \setminus A^{(0)} = g(B)$. Further, it is a countable equidecomposition. To prove this, conside the partitions $A = \cup_{i=1}^{\infty} A_i$ and $B = \cup_{i=1}^{\infty} B_i$, with respect to which f and g are (respectively) equidecompositions with translation parameters $\{t_i\}$ and $\{s_i\}$. Then

$$A = \Big\{ \bigcup_{i=1}^{\infty} (A_i \cap A^{(*)}) \Big\} \bigcup \Big\{ \bigcup_{i=1}^{\infty} (g(B_i) \cap (A \setminus A^{(*)})) \Big\} \tag{2}$$

is a countable partition of A and it is easy to check that h is an equidecomposition with respect to this partition. Indeed $h|_{A_i \cap A^{(*)}} = f|_{A_i \cap A^{(*)}} = R_{t_i}|_{A_i \cap A^{(*)}}$ and $h|_{g(B_i)\cap(A\setminus A^{(*)})} = g^{-1}|_{g(B_i)\cap(A\setminus A^{(*)})} = R_{-s_i}|_{g(B_i)\cap(A\setminus A^{(*)})}$.

It remains to prove that h is bijective. To this end, define the mapping $l : B \to A$ by

$$l(y) = \begin{cases} f^{-1}(y) & \text{if } g(y) \in A^{(*)}, \\ g(y) & \text{otherwise.} \end{cases} \tag{3}$$

It is not hard to prove that l is the inverse of h. Indeed, if $x \equiv g(y) \notin A^{(*)}$, then $h(x) = g^{-1}(x) = y$. On the other hand, if $g(y) \in A^{(*)}$, then $g(y) = (g \circ f)^k(A^{(0)})$ for some $k \geq 1$ (because $A^{(0)} \cap g(B) = \emptyset$). By injectivity of g, $y = f((g \circ f)^{k-1}(x_0))$ for some $x_0 \in A^{(0)}$. If we let $x = l(y) = f^{-1}(y)$, then $x \in A^{(*)}$ as well, whence $h(x) = f(x) = y$.

B1

Consider the equivalence relation $x \sim y$ if $(x - y) \in \mathbb{Q}$, and let $\mathcal{E}$ denote the collection of its equivalence classes. For every $E \in \mathcal{E}$, $E \cap [0, 1/2]$ is non-empty and by the axiom of choice, there exist a choice function $E \mapsto x_E$ such that $x_E \in E \cap [0, 1/2]$ for each E. Obviously $E = x_E + \mathbb{Q}$. Therefore

$$\mathbb{R} = \cup_{E \in \mathcal{E}} E = \cup_{E \in \mathcal{E}} \{x_E + \mathbb{Q}\} = \cup_{x \in C} \{x + \mathbb{Q}\}\,, \tag{4}$$

where $C = \{x_E\}_{E \in \mathcal{E}}$.

B2

Since the rationals are countable, there exist partitions in isolated points

$$\mathbb{Q} \cap [0, 1/2] = \cup_{i=1}^{\infty} \{q_i\}\,, \qquad \mathbb{Q} = \cup_{i=1}^{\infty} \{p_i\}\,. \tag{5}$$

Of course, for any i, $\{p_i\} = R_{t_i}(\{q_i\})$ if we set $t_i = p_i - q_i$.

B3

Using the enumerations of $\mathbb{Q} \cap [0, 1/2]$ and of $\mathbb{Q}$ at the previous point, we get

$$A \equiv \cup_{x \in C} \{x + (\mathbb{Q} \cap [0, 1/2])\} = \cup_{i=1}^{\infty} \{q_i + C\}\,, \qquad \mathbb{R} = \cup_{x \in C} \{x + \mathbb{Q}\} = \cup_{i=1}^{\infty} \{p_i + C\}\,. \tag{6}$$

We have $\{p_i + C\} = R_{t_i}(\{q_i + C\})$ if we set $t_i = p_i - q_i$ as above.

B4

Clearly $[0,1]$ is equidecomposable with $[0,1] \subseteq \mathbb{R}$ (via the identity mapping). On the other hand $\mathbb{R}$ is equidecomposable with $A \subseteq [0,1]$. By points A1, A2, this implies that $[0,1]$ is equidecomposable with $\mathbb{R}$.

This implies that there exists no measure on $\mathbb{R}$ satisfying the following requirements

1. The measure is countably additive (As it should be).
2. $\mu([0,1]) \notin \{0, \infty\}$.
3. Any set is measurable.
4. $\mu(S) = \mu(R_t S)$ for any measurable set S.