Stat 310A/Math 230A Theory of Probability

Homework 5

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Due on November 6, 2019

- **(Important)** Return all assignments with a cover page stapled to the front of the work that details only basic information: name, quarter, course.
- Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) for each problem.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.
- In some cases, multiple homework options will be proposed (and indicated as 'Option 1', 'Option 2', etc.). You are welcome to work on all the problems proposed (solutions will be posted), but should submit only those corresponding to one 'Option'.

Exercises on the law of large numbers and Borel-Cantelli

Solve Exercises [2.1.5], [2.1.13], [2.2.9], [2.2.26], [2.3.14] (I also suggest to read the illustration in example 2.2.27) in Amir Dembo's lecture notes.

Solutions will be provided for exercises [2.3.9] and [2.2.24] (do not return these!).

Optional: An exercise on Markov chains

Let $\mathcal{X}$ be a finite set, q be a probability law over $\mathcal{X}$ and $p = \{p(x,y)\}_{x,y\in\mathcal{X}}$ a $|\mathcal{X}|\times|\mathcal{X}|$ matrix of non-negative numbers such that, for any $x \in \mathcal{X}$:

$$\sum_{y\in\mathcal{X}} p(x,y) = 1\,. \tag{1}$$

A Markov chain is a probability measure over $(\mathcal{X}^{\mathbb{N}},\mathcal{F})$, with $\mathcal{X}^{\mathbb{N}} = \{(\omega_0,\omega_1,\omega_2,\dots) : \omega_i \in \mathcal{X}\}$ and $\mathcal{F}$ the σ-algebra generated by cylindrical sets, such that

$$\mathbb{P}\big(\{\omega : \omega_0 = x_0,\, \omega_1 = x_1 \dots \omega_n = x_n\}\big) = q(x_0)\prod_{i=0}^{n-1} p(x_i,x_{i+1})\,, \tag{2}$$

for any $n \geq 0$. Check that this indeed defines a probability distribution using Kolmogorov extension theorem.

Let $X_i(\omega) = \omega_i$ and recall that the tail σ-algebra is defined by

$$\mathcal{T} \equiv \bigcap_{n\geq 0} \sigma(\{X_i\}_{i\geq n})\,. \tag{3}$$

Prove that, if $p(x,y) > 0$ for any $x,y\in\mathcal{X}$, then $\mathcal{T}$ is trivial.

[Hint: It might be a good idea to remind yourself the statement of Perron-Frobenius theorem]