Stat 310A/Math 230A Theory of Probability

Homework 6 Solutions

Andrea Montanari *Due on November 15, 2017*

Exercises on the strong law of large numbers and the central limit theorem

Exercise [2.3.13]

Clearly, $|X_n| = |X_{n-1}||U_n|$, resulting with

$$\log|X_n| = \sum_{k=1}^{n} \log|U_k| + \log|X_0|\,.$$

As $\mathbf{P}(|U_1| \le r) = r^2$ for $0 \le r \le 1$, it follows from Corollary 1.3.60 and integration by parts that $\mathbf{E}\log|U_1| = \int_0^1 2r\log r dr = -1/2$. Further, $\log|U_k|$ are i.i.d. so by the strong law of large numbers we have that $n^{-1}\log|X_n| \overset{a.s.}{\to} -1/2$.

Exercise [2.3.22]

Let $X_k \equiv Y_k/[k(\log k)^{1+\epsilon}]^{1/2}$. By Kronecker's Lemma it is suffcient to prove that the series $\sum_{k=1}^{\infty} X_k$ converges almost surely. For this we use Theorem 2.3.16 by noting that $\mathbb{E}(X_k) = 0$ and

$$\sum_{k=1}^{\infty} \mathrm{Var}(X_k) \le B \sum_{k=1}^{\infty} \frac{1}{k(\log k)^{1+\epsilon}} < \infty\,. \tag{1}$$

Exercise [2.3.24]

Let $Y_n = X_n I_{\{X_n<1\}}$ and $a_n = \mathbf{E}Y_n$, both in $[0,1)$.

1. The assumed finiteness of the series whose terms are the sum of two non-negative quantities, implies that each of the corresponding series is finite. That is, both $\sum_n \mathbf{P}(X_n \ge 1) < \infty$ and $\sum_n a_n < \infty$. By the first Borel-Cantelli lemma, the finiteness of $\sum_n \mathbf{P}(X_n \ge 1)$ implies that $\mathbf{P}(X_n \ge 1 \text{ i.o. }) = 0$. Thus, to prove that $\sum_n X_n(\omega)$ converges w.p.1, it suffices to prove that the series $\sum_n Y_n(\omega)$ converges w.p.1. Since $Y_n \in [0,1)$, it follows that $Y_n^2 \le Y_n$, hence $\mathrm{Var}(Y_n) \le \mathbf{E}Y_n^2 \le a_n$. Consequently, the finiteness of $\sum_n a_n$ implies that $\sum_n \mathrm{Var}(Y_n) < \infty$. Theorem 2.3.16 then results with the convergence w.p.1. of the random series $\sum_n (Y_n(\omega) - a_n)$. Since we know that the non-negative constant $\sum_n a_n$ is finite, this implies that the random series $\sum_n Y_n(\omega)$ also converges w.p.1.

2. We prove the converse by proving the contrapositive. If $\sum_n \mathbf{P}(X_n \ge 1)$ is infinite, then with $\{X_n\}$ independent, by the second Borel-Cantelli lemma we know that $\mathbf{P}(X_i \ge 1 \text{ i.o. }) = 1$, which implies that $\sum_n X_n(\omega)$ diverges w.p.1. Suppose next that $\sum_n a_n$ is infinite. Then, by the hint, $\prod_n (1 - a_n) = 0$, or equivalently, $e_k = \prod_{n=1}^{k}(1 - a_n) \downarrow 0$ as $k \to \infty$. Since Y_n are independent, we have that $e_k = \mathbf{E}Z_k$ for the non-negative random variable $Z_k = \prod_{n=1}^{k}(1 - Y_n) \le 1$. Further, $Z_k \downarrow Z_\infty = \prod_n (1 - Y_n) \ge 0$ for $k \to \infty$ and any $\omega \in \Omega$. By the bounded convergence theorem this implies that $e_k \to \mathbf{E}Z_\infty$. Consequently, $\mathbf{E}Z_\infty = 0$, hence also $Z_\infty = \prod_n (1 - Y_n) = 0$ w.p.1. Applying the hint in the converse direction we conclude that $\sum_n X_n \ge \sum_n Y_n = \infty$ w.p.1.

3. The series $S := \sum_n G_n^2$ of non-negative terms converges in $\overline{\mathbb{R}}$ so the question is merely when is $\mathbf{P}(S(\omega) < \infty) = 1$. Since $\mathbf{E}G_n^2 = \mu_n^2 + v_n$ for all n, we have that $e = \mathbf{E}S$. Consequently, if e is finite then $\mathbf{P}(S < \infty) = 1$. Conversely, assuming $\mathbf{P}(S < \infty) = 1$, upon applying part (b) for $X_n = G_n^2$ we find that $s := \sum_n \mathbf{E}[\min(G_n^2, 1)]$ must be finite. As $G_n \stackrel{\mathcal{D}}{=} \mu_n + \sqrt{v_n} Y$ for Y of a standard normal distribution, we deduce by linearity of the expectation that $s = \mathbf{E}f(Y)$. With $Y \stackrel{\mathcal{D}}{=} -Y$ we further find that $s = \frac{1}{2}\mathbf{E}[f(Y) + f(-Y)]$. Now, by the hint provided, $f(y) + f(-y) = \infty$ for all $y \neq 0$ in case $e = \infty$. In particular, if $e = \infty$ then also $s = \infty$, contradicting our assumption that $\mathbf{P}(S < \infty) = 1$ and thus proving our thesis.

4. Observe that $\mathbf{E}\tau_n = 1/\lambda_n$, so that if $\sum_n 1/\lambda_n < \infty$, then by the non-negativity of τ_n and MCT, we have $\mathbf{E}\sum_n \tau_n < \infty$, and thus $\sum_n \tau_n < \infty$ a.s. For the converse, suppose $\sum_n \tau_n < \infty$ a.s. Then by part (a), we have
$$\sum_n \mathbb{P}(\tau_n \geq 1) + \mathbf{E}\tau_n I(\tau_n < 1) < \infty.$$
By direct computation, we have
$$\mathbb{P}(\tau_n \geq 1) = e^{-\lambda_n},$$
$$\mathbf{E}\tau_n I(\tau_n < 1) = \frac{1}{\lambda_n}(1 - (\lambda_n + 1)e^{-\lambda_n}).$$
To finish, observe that if $\sum_n \lambda_n^{-1}(1 - e^{-\lambda_n}) < \infty$, then $\sum \lambda_n^{-1} < \infty$.

Exercise [3.1.11]

1. We apply Lindeberg's CLT to the sum $\widehat{S}_n$ of the zero mean, mutually independent variables $X_{n,k} = v_n^{-1/2}(X_k - \mathbf{E}X_k)$. Since $\widehat{S}_n$ is then of unit variance, it suffices to check Lindeberg's condition
$$g_n(\varepsilon) = \sum_{k=1}^n \mathbf{E}[X_{n,k}^2; |X_{n,k}| \geq \varepsilon] = v_n^{-1} \sum_{k=1}^n \mathbf{E}[(X_k - \mathbf{E}X_k)^2; |X_k - \mathbf{E}X_k| \geq \varepsilon v_n^{1/2}]$$
$$\leq \varepsilon^{2-q} v_n^{-q/2} \sum_{k=1}^n \mathbf{E}[|X_k - \mathbf{E}X_k|^q] \to 0$$
in order to conclude with the stated CLT.

2. We have $v_n = n$ and for some $q > 2$,
$$v_n^{-q/2} \sum_{k=1}^n \mathbf{E}(|X_k - \mathbf{E}X_k|^q) \leq v_n^{-q/2} \sum_{k=1}^n C = Cn^{1-q/2} \to 0$$
as $n \to \infty$. The stated convergence in distribution thus follows from Lyapunov's theorem.

3. Let $a_k = 2^k$, and let $X_k = \pm a_k$ w.p. $1/2$. Observe $\mathbf{E}X_k = 0, \mathrm{Var}(X_k) = a_k^2$, and $\mathbf{E}|X_k|^q = a_k^q$. Thus for any q, we have $\mathbf{E}|X_k|^q = \mathrm{Var}(X_k)^{q/2}$. Observe that $v_n \geq \mathrm{Var}(X_n) = 2^{2n}$. This implies that $|S_n/\sqrt{v_n}| < 100$ (say), which implies $\mathbb{P}(S_n/\sqrt{v_n} \in (-100, 100]) = 1$ for all n. Thus $S_n/\sqrt{v_n}$ cannot converge in distribution to a standard Normal.

Exercise [3.1.10]

1. By independence,
$$b_n = \mathrm{Var}(R_n) = \sum_{k=1}^n \mathrm{Var}(B_k) = \sum_{k=1}^n k^{-1}(1 - k^{-1}) = \sum_{k=1}^n k^{-1} - \sum_{k=1}^n k^{-2}\,.$$

Further, since $\log n = \int_1^n x^{-1} dx$, it follows from the monotonicity of $x \mapsto x^{-1}$ that $\sum_{k=2}^n k^{-1} \leq \log n \leq \sum_{k=1}^n k^{-1}$. With $\sum_k k^{-2}$ finite and $\log n \to \infty$, we get that $b_n / \log n \to 1$ as claimed.

2. Since $|X_{n,k}| \leq (\log n)^{-1/2}$ for all n, k and ω, it follows that $g_n(\varepsilon)$ of (3.1.4) is zero as soon as $n > \exp(\varepsilon^{-2})$, so Lindeberg's condition is satisfied here. Further, by part (a) the zero-mean random variables $X_{n,k}$ are such that $v_n = \sum_{k=1}^n \mathbf{E} X_{n,k}^2 = b_n / \log n \to 1$ as $n \to \infty$.

3. Applying Lindeberg's CLT we have that $(R_n - \mathbf{E} R_n)/\sqrt{\log n} \xrightarrow{\mathcal{D}} G$. It is easy to check that such convergence in distribution remains in effect even after adding the non-random $(\mathbf{E} R_n - \log n)/\sqrt{\log n} \to 0$.