Exercises on the strong law of large numbers and the central limit theorem

Exercise [2.3.13]

Clearly, \( |X_n| = |X_{n-1}| |U_n| \), resulting with

\[
\log |X_n| = \sum_{k=1}^{n} \log |U_k| + \log |X_0|.
\]

As \( P(|U_1| \leq r) = r^2 \) for \( 0 \leq r \leq 1 \), it follows from Corollary 1.3.60 and integration by parts that \( E \log |U_1| = \int_{0}^{1} 2r \log r dr = -1/2 \). Further, \( \log |U_k| \) are i.i.d. so by the strong law of large numbers we have that \( n^{-1} \log |X_n| \rightarrow -1/2 \).

Exercise [2.3.22]

Let \( X_k = Y_k/[k(\log k)^{1+\epsilon}] \). By Kronecker’s Lemma it is sufficient to prove that the series \( \sum_{k=1}^{\infty} X_k \) converges almost surely. For this we use Theorem 2.3.16 by noting that \( E(X_k) = 0 \) and

\[
\sum_{k=1}^{\infty} \text{Var}(X_k) \leq B \sum_{k=1}^{\infty} \frac{1}{k(\log k)^{1+\epsilon}} < \infty.
\]

Exercise [2.3.24]

Let \( Y_n = X_n I_{\{X_n < 1\}} \) and \( a_n = EY_n \), both in \([0,1)\).

1. The assumed finiteness of the series whose terms are the sum of two non-negative quantities, implies that each of the corresponding series is finite. That is, both \( \sum_{n} P(X_n \geq 1) < \infty \) and \( \sum_{n} a_n < \infty \). By the first Borel-Cantelli lemma, the finiteness of \( \sum_{n} P(X_n \geq 1) \) implies that \( P(X_n \geq 1 \text{ i.o.}) = 0 \). Thus, to prove that \( \sum_{n} X_n(\omega) \) converges w.p.1, it suffices to prove that the series \( \sum_{n} Y_n(\omega) \) converges w.p.1. Since \( Y_n \in [0,1] \), it follows that \( Y_n^2 \leq Y_n \), hence \( \text{Var}(Y_n) \leq EY_n^2 \leq a_n \). Consequently, the finiteness of \( \sum_{n} a_n \) implies that \( \sum_{n} \text{Var}(Y_n) < \infty \). Theorem 2.3.16 then results with the convergence w.p.1. of the random series \( \sum_{n} (Y_n(\omega) - a_n) \). Since we know that the non-negative constant \( \sum_{n} a_n \) is finite, this implies that the random series \( \sum_{n} Y_n(\omega) \) also converges w.p.1.

2. We prove the converse by proving the contrapositive. If \( \sum_{n} P(X_n \geq 1) \) is infinite, then with \( \{X_n\} \) independent, by the second Borel-Cantelli lemma we know that \( P(X_i \geq 1 \text{ i.o.}) = 1 \), which implies that \( \sum_{n} X_n(\omega) \) diverges w.p.1. Suppose next that \( \sum_{n} a_n \) is infinite. Then, by the hint, \( \prod_{n}(1-a_n) = 0 \), or equivalently, \( e_k = \prod_{n=1}^{k}(1-a_n) \downarrow 0 \) as \( k \rightarrow \infty \). Since \( Y_n \) are independent, we have that \( e_k = E Z_k \) for the non-negative random variable \( Z_k = \prod_{n=1}^{k}(1-Y_n) \leq 1 \). Further, \( Z_k \downarrow Z_{\infty} = \prod_{n}(1-Y_n) \geq 0 \) for \( k \rightarrow \infty \) and any \( \omega \in \Omega \). By the bounded convergence theorem this implies that \( e_k \rightarrow E Z_{\infty} \). Consequently, \( E Z_{\infty} = 0 \), hence also \( Z_{\infty} = \prod_{n}(1-Y_n) = 0 \) w.p.1. Applying the hint in the converse direction we conclude that \( \sum_{n} X_n \geq \sum_{n} Y_n = \infty \) w.p.1.
3. The series $S := \sum_n G_n^2$ of non-negative terms converges in $\mathbb{R}$ so the question is merely when is $\mathbb{P}(S(\omega) < \infty) = 1$. Since $\mathbb{E}G_n^2 = \mu_n^2 + v_n$ for all $n$, we have that $e = \mathbb{E}S$. Consequently, if $e$ is finite then $\mathbb{P}(S < \infty) = 1$. Conversely, assuming $\mathbb{P}(S < \infty) = 1$, upon applying part (b) for $X_n = G_n^2$ we find that $s := \sum_n \mathbb{E}\min(G_n^2, 1)$ must be finite. As $G_n \overset{D}{=} \mu_n + \sqrt{v_n}Y$ for $Y$ of a standard normal distribution, we deduce by linearity of the expectation that $s = \mathbb{E}f(Y)$. With $Y \overset{D}{=} -Y$ we further find that $s = \frac{1}{2}\mathbb{E}[f(Y) + f(-Y)]$. Now, by the hint provided, $f(y) + f(-y) = \infty$ for all $y \neq 0$ in case $e = \infty$. In particular, if $e = \infty$ then also $s = \infty$, contradicting our assumption that $\mathbb{P}(S < \infty) = 1$ and thus proving our thesis.

4. Observe that $\mathbb{E}\tau_n = 1/\lambda_n$, so that if $\sum_n 1/\lambda_n < \infty$, then by the non-negativity of $\tau_n$ and MCT, we have $\mathbb{E}\sum_n \tau_n < \infty$, and thus $\sum_n \tau_n < \infty$ a.s. For the converse, suppose $\sum_n \tau_n < \infty$ a.s. Then by part (a), we have

$$\sum_n \mathbb{P}(\tau_n \geq 1) + \mathbb{E}\tau_n I(\tau_n < 1) < \infty.$$ 

By direct computation, we have

$$\mathbb{P}(\tau_n \geq 1) = e^{-\lambda_n},$$ 

$$\mathbb{E}\tau_n I(\tau_n < 1) = \frac{1}{\lambda_n} (1 - (\lambda_n + 1)e^{-\lambda_n}).$$

To finish, observe that if $\sum_n \lambda_n^{-1}(1 - e^{-\lambda_n}) < \infty$, then $\sum \lambda_n^{-1} < \infty$.

**Exercise [3.1.11]**

1. We apply Lindeberg’s CLT to the sum $\hat{S}_n$ of the zero mean, mutually independent variables $X_{n,k} = v_n^{-1/2}(X_k - \mathbb{E}X_k)$. Since $\hat{S}_n$ is then of unit variance, it suffices to check Lindeberg’s condition

$$g_n(\varepsilon) = \sum_{k=1}^{n} \mathbb{E}[X_{n,k}^2; |X_{n,k}| \geq \varepsilon] = v_n^{-1} \sum_{k=1}^{n} \mathbb{E}[(X_k - \mathbb{E}X_k)^2; |X_k - \mathbb{E}X_k| \geq \varepsilon v_n^{1/2}]$$

$$\leq \varepsilon^2 v_n^{-q/2} \sum_{k=1}^{n} \mathbb{E}[|X_k - \mathbb{E}X_k|^q] \to 0$$

in order to conclude with the stated CLT.

2. We have $v_n = n$ and for some $q > 2$,

$$v_n^{-q/2} \sum_{k=1}^{n} \mathbb{E}[(X_k - \mathbb{E}X_k)^q] \leq v_n^{-q/2} \sum_{k=1}^{n} C = C n^{1-q/2} \to 0$$

as $n \to \infty$. The stated convergence in distribution thus follows from Lyapunov’s theorem.

3. Let $a_k = 2^k$, and let $X_k = \pm a_k$ w.p. $1/2$. Observe $\mathbb{E}X_k = 0$, $\text{Var}(X_k) = a_k^2$, and $\mathbb{E}|X_k|^q = a_k^q$. Thus for any $q$, we have $\mathbb{E}|X_k|^q = \text{Var}(X_k)^{q/2}$. Observe that $v_n \geq \text{Var}(X_n) = 2^{2n}$. This implies that $|S_n/\sqrt{v_n}| < 100$ (say), which implies $\mathbb{P}(S_n/\sqrt{v_n} \in (-100, 100)) = 1$ for all $n$. Thus $S_n/\sqrt{v_n}$ cannot converge in distribution to a standard Normal.

**Exercise [3.1.10]**

1. By independence,

$$b_n = \text{Var}(R_n) = \sum_{k=1}^{n} \text{Var}(B_k) = \sum_{k=1}^{n} k^{-1}(1 - k^{-1}) = \sum_{k=1}^{n} k^{-1} - \sum_{k=1}^{n} k^{-2}.$$
Further, since \( \log n = \int_1^n x^{-1} \, dx \), it follows from the monotonicity of \( x \mapsto x^{-1} \) that \( \sum_{k=2}^{n} k^{-1} \leq \log n \leq \sum_{k=1}^{n} k^{-1} \). With \( \sum_{k} k^{-2} \) finite and \( \log n \to \infty \), we get that \( b_n / \log n \to 1 \) as claimed.

2. Since \( |X_{n,k}| \leq (\log n)^{-1/2} \) for all \( n, k \) and \( \omega \), it follows that \( g_n(\varepsilon) \) of (3.1.4) is zero as soon as \( n > \exp(\varepsilon^{-2}) \), so Lindeberg’s condition is satisfied here. Further, by part (a) the zero-mean random variables \( X_{n,k} \) are such that \( v_n = \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2 = b_n / \log n \to 1 \) as \( n \to \infty \).

3. Applying Lindeberg’s CLT we have that \( (R_n - \mathbb{E}R_n) / \sqrt{\log n} \overset{D}{\to} G \). It is easy to check that such convergence in distribution remains in effect even after adding the non-random \( (\mathbb{E}R_n - \log n) / \sqrt{\log n} \to 0 \).