Stat 310A/Math 230A Theory of Probability

Homework 8

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- **(Important)** Return all assignments with a cover page stapled to the front of the work that details only basic information: name, quarter, course.
- Solutions should be complete and concisely written.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

Exercises on characteristic functions

Solve Exercises [3.3.10], [3.3.21], [3.3.23], in Amir Dembo's lecture notes.

An exercise on weak convergence of measures

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the set of (infinite) binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$, and consider the topolgy generated by the following basis of neighborhoods of $\omega \in \Omega$:

$$N_\ell(\omega) = \left\{\xi \in \Omega \,:\, \xi_1^\ell = \omega_1^\ell\right\}, \tag{1}$$

with $\ell \in \mathbb{N}$ (here we use the notation $\omega_1^\ell = (\omega_1, \dots, \omega_\ell)$). Let $\mathcal{B}_\Omega$ be the Borel σ-algebra associated to this topology.

For n even, let A_n be the set of sequences defined as follows

$$A_n = \left\{\omega \in \Omega \,:\, \sum_{i=1}^{n} \omega_i = n/2,\ \omega_i = 0 \text{ for all } i > n\right\}. \tag{2}$$

Consider the sequence of probability measures $\{\nu_n\}_{n\in 2\mathbb{N}}$, with ν_n the uniform distribution over A_n, i.e.

$$\nu_n(\{\omega\}) = \begin{cases} \binom{n}{n/2}^{-1} & \text{if } \omega \in A_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

1. Show that, for each n, ν_n is indeed a measure over $\mathcal{B}_\Omega$.
2. What is the weak limit of ν_n as $n \to \infty$? Prove your answer.

In solving the last point you can assume the following

Fact 1. *Let $h : \Omega \to \mathbb{R}$ be a continuous function. Then h is uniformly continuous in the following sense. There exists a function $\delta : \mathbb{N} \to \mathbb{R}$, $\ell \mapsto \delta(\ell)$, with $\lim_{\ell\to\infty} \delta(\ell) = 0$, such that, for any $\omega \in \Omega$, $\omega' \in N_\ell(\omega)$, we have $|h(\omega') - h(\omega)| \le \delta(\ell)$.*