1 Characterization of distribution functions

In this section, we give necessary and sufficient conditions for a function \( F : \mathbb{R} \rightarrow [0, 1] \) to be a distribution function.

**Theorem 1** *(Thm 1.2.36, Dembo’s Notes)*. A function \( F : \mathbb{R} \rightarrow [0, 1] \) is a distribution function of some R.V. if and only if

(a) \( F \) is non-decreasing;

(b) \( \lim_{x \to \infty} F(x) = 1 \) and \( \lim_{x \to -\infty} F(x) = 0 \);

(c) \( F \) is right-continuous, i.e. \( \lim_{y \downarrow x} F(y) = F(x) \).

**Proof** “\( \Rightarrow \)”. Let \( F \) be the distribution of some random variable \( X \) on a probability space \( (\Omega, \mathcal{F}, P) \). Let \( x \leq y \), then \( \{ \omega : X(\omega) \leq x \} \subseteq \{ \omega : X(\omega) \leq y \} \), hence \( F(x) = \mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y) = F(y) \). By continuity of \( P \), we have

\[
\lim_{x \to \infty} F(x) = \lim_{x \to \infty} P(\{ \omega : X(\omega) \leq x \}) = P(\lim_{x \to \infty} \{ \omega : X(\omega) \leq x \}) = P(\Omega) = 1
\]

and similarly \( \lim_{x \to 0} F(x) = P(\emptyset) = 0 \). Take \( x \in \mathbb{R} \), we have

\[
\lim_{y \downarrow x} \{ \omega : X(\omega) \leq y \} = \{ \omega : X(\omega) \leq x \}.
\]

Hence by continuity of \( P \),

\[
\lim_{y \downarrow x} F(y) = \lim_{y \downarrow x} P(\{ \omega : X(\omega) \leq y \}) = P(\{ \omega : X(\omega) \leq x \}) = F(x).
\]

“\( \Leftarrow \)”. We define \( X^- (\omega) = \sup \{ y : F(y) < \omega \} \) on the probability space \( ((0,1], \mathcal{B}_{(0,1]}, U) \), i.e. \( (0, 1] \) with the uniform distribution. Note that for all \( \omega \in (0, 1) \), as \( F \) is non-decreasing and its range contains \( (0, 1) \), the set \( \{ y : F(y) \leq \omega \} \) is non-empty and has a finite upper bound. Hence \( X^- : (0, 1) \to \mathbb{R} \) is well-defined.

We are going to show that the distribution function of \( X^- \) equals to \( F \). We claim that for all \( x \in \mathbb{R} \),

\[
\{ \omega : X^- (\omega) \leq x \} = \{ \omega : \omega \leq F(x) \}.
\]

This implies that the LHS is in \( \mathcal{B}_{(0,1]} \) and that

\[
U(\{ \omega : X^- (\omega) \leq x \}) = U(\{ \omega : \omega \leq F(x) \}) = U((0, F(x)]) = F(x),
\]

so the distribution function of \( X^- (\omega) \) is \( F \).
It remains to show \(1\). Suppose \(F(x) \geq \omega\), then by monotonicity \(x \geq y\) for all \(y\) such that \(F(y) < \omega\), giving that \(X^-(\omega) = \sup \{y : F(y) < \omega\} \leq x\). Conversely, suppose \(X^-(\omega) \leq x\), we claim that \(F(x) \geq \omega\) has to be true. If not, then \(F(x) < \omega\). By the right continuity of \(F\), there exists some \(\varepsilon > 0\) such that \(F(x + \varepsilon) < \omega\), giving that

\[
X^-(\omega) = \sup \{y : F(y) < \omega\} \geq x + \varepsilon > x,
\]
a contradiction. Hence, we must have \(F(x) \geq \omega\).

\(\square\)

2 Completion of measure spaces

A nice property about the Lebesgue measure is the following: any subset of a measure-zero set is measurable. To see this, for example on \(\mathbb{R}\), let \(A\) have measure zero and \(B \subset A\). For any \(E \subset \mathbb{R}\), we have

\[
P^*(E \cap B) + P^*(E \cap B^c) \\
\leq P^*(E \cap A) + P^*(E \cap B^c \cap A^c) + P^*(E \cap B^c \cap A) \\
= P^*(E \cap A) + P^*(E \cap A^c) + P^*(E \cap (A \setminus B)) = P^*(E \cap A) + P^*(E \cap A^c) = P^*(E).
\]

The last equality follows as \(E \cap (A \setminus B)\) is a subset of \(A\), so \(P^*(E \cap (A \setminus B)) \leq P^*(A) = 0\). Hence, \(B\) is measurable, and \(P(B) \leq P(A) = 0\).

However, such a property might not be present in a general measure space. We are going present a result saying that one can always slightly enlarge the \(\sigma\)-algebra and extend the measure to get this property.

**Definition 1** (Def 1.1.34, Dembo's Notes). We say that a measure space \((\Omega, \mathcal{F}, \mu)\) is complete if any subset \(N\) of any \(B \in \mathcal{F}\) with \(\mu(B) = 0\) is also in \(\mathcal{F}\).

**Theorem 2** (Thm 1.1.35, Dembo's Notes). Given a measure space \((\Omega, \mathcal{F}, \mu)\), let

\[
\mathcal{N} = \{N : N \subseteq A \text{ for some } A \in \mathcal{F} \text{ with } \mu(A) = 0\}
\]
denote the collection of \(\mu\)-null sets. Then, there exists a complete measure space \((\Omega, \overline{\mathcal{F}}, \overline{\mu})\), called the completion of the measure space \((\Omega, \mathcal{F}, \mu)\), such that \(\overline{\mathcal{F}} = \{F \cup N : F \in \mathcal{F}, N \in \mathcal{N}\}\) and \(\overline{\mu} = \mu\) on \(\mathcal{F}\).

Intuitively the result is quite expected: we can add all the \(\mu\)-null sets into \(\mathcal{F}\) and let them have measure zero.

**Proof** We divide the proof into the following steps.

(1) \(\overline{\mathcal{F}}\) is a \(\sigma\)-algebra.

Clearly \(0 \in \overline{\mathcal{F}}\). Take any \(B \in \overline{\mathcal{F}}\), then \(B = F \cup N\) with \(F \in \mathcal{F}\) and \(N \in \mathcal{N}\). In particular, there exists \(A \in \mathcal{F}\) such that \(\mu(A) = 0\) and \(N \subseteq A\). Thus

\[
B^c = F^c \cap N^c = ((F^c \cap A) \cap N^c) \cup ((F^c \cap A^c) \cap N^c) = (F^c \cap A^c) \cup (F^c \cap A \cap N^c).
\]

As \(F^c \cap A^c \in \mathcal{F}\) and \(F^c \cap A \cap N^c \subseteq A\), we have \(B^c \in \overline{\mathcal{F}}\). For any \(\{B_n\} \in \overline{\mathcal{F}}\), let \(B_n = F_n \cup N_n\) and \(N_n \subseteq A_n\) be their decompositions, then

\[
\bigcup_n B_n = \left(\bigcup_n F_n\right) \cup \left(\bigcup_n N_n\right).
\]
As $\bigcup_n N_n \subseteq \bigcup_n A_n$ and $\bigcup_n A_n \in \mathcal{F}$ with $\mu(\bigcup_n A_n) = \sum_n \mu(A_n) = 0$, we have $\bigcup_n N_n \in \mathcal{N}$ and thus $\bigcup_n B_n \in \mathcal{F}$.

(2) Define $\mu(B) = \mu(F)$ for $B = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$. $\mu$ is well defined.

We need to verify that if $B$ have two decompositions $B = F_1 \cup N_1$ and $B = F_2 \cup N_2$, then $\mu(F_1) = \mu(F_2)$. Indeed, we have

$$F_1 \subseteq F_1 \cup N_1 = B = F_2 \cup N_2 \subseteq F_2 \cup A_2,$$

where $\mu(A_2) = 0$. Hence $\mu(F_1) \leq \mu(F_2 \cup A_2) \leq \mu(F_2) + \mu(A_2) = \mu(F_2)$. That $\mu(F_2) \leq \mu(F_1)$ follows by exchanging the roles of $F_1$ and $F_2$.

(3) $\mu$ is a measure on $\mathcal{F}$ and agrees with $\mu$ on $\mathcal{F}$.

Clearly $\mu(\emptyset) = 0$. Let $\{B_n\}$ be a sequence of disjoint sets in $\mathcal{F}$ with decompositions $B_n = F_n \cup N_n$. As $F_n$ and $N_n$ are all disjoint, we have

$$\mu\left(\bigcup_n B_n\right) = \mu\left(\bigcup_n F_n \cup \bigcup_n N_n\right) = \mu\left(\bigcup_n F_n\right) = \sum_n \mu(F_n) = \sum_n \mu(B_n).$$

Hence $\mu$ is countably additive. For any $F \in \mathcal{F}$, $F = F \cup \emptyset$, so $\mu(F) = \mu(F)$.

(4) $(\Omega, \mathcal{F}, \mu)$ is complete.

Take any $B \in \mathcal{F}$ with $\mu(B) = 0$. Then $B = F \cup N$ for some $F \in \mathcal{F}$ and $N \in \mathcal{N}$. We have $\mu(F) = \mu(F) \leq \mu(B) = 0$, so $F$ itself is a measure-zero set in $\mathcal{F}$, hence $B \in \mathcal{N}$. ($A \cup F$ contains $B$ for $A$ containing $N$.) So if $C \subseteq B$, then $C \in \mathcal{N}$, and thus $C \in \mathcal{F}$.

Remark Another way of constructing the completion is to look at the outer measure $\mu^*$ on $\mu^*$-measurable sets $\mathcal{G}$. One can show that this construction coincides with our construction, in particular, $\mathcal{G} = \mathcal{F}$. (See Exercise 3.10(c) in Billingsley [1].)

References