1 The Law of iterated logarithm

In this note we prove the law of iterated logarithm, mainly following [? , Chapter 9]. Let $X_i$ be independent R.V.-s with mean 0 and variance 1. The central limit theorem characterizes the behavior of $S_n = X_1 + \cdots + X_n$ and states that $S_n = O_p(\sqrt{n})$. The law of iterated algorithm refines this result dramatically, precisely characterizing the scalings of the extrema of $S_n$.

**Theorem 1** (Law of iterated logarithm). We have

$$P\left(\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1\right) = 1.$$  

Equivalently, the theorem states the following: for all $\varepsilon > 0$,

$$P\left(S_n \geq (1 + \varepsilon)\sqrt{2n \log \log n}\right) = 0,$$  

$$P\left(S_n \geq (1 - \varepsilon)\sqrt{2n \log \log n}\right) = 1.$$  

Hence, showing LIL requires estimating the probability $P(S_n/\sqrt{n} \geq t)$ very accurately, for $t$ on the order of $\sqrt{\log \log n}$. The following lemma presents such a result.

**Lemma 1.1.** Let $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$, then

$$P\left(\frac{S_n}{\sqrt{n}} \geq a_n\right) = \exp\left(-\frac{1}{2}a_n^2(1 + \xi_n)\right),$$

where $\xi_n \to 0$.

We will also need a variant of Kolmogorov’s maximal inequality. Let $M_n = \max_{1 \leq k \leq n} S_k$ be the maximum process.

**Lemma 1.2.** For $\alpha \geq \sqrt{2}$, we have

$$P\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq 2P\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right).$$

**Proof of Theorem ??** We prove the result by looking at a subsequence $S_{n_k}$ where $n_k = \theta^k$ for some carefully chosen $\theta > 1$. We bound the deviation probability carefully and use Borel-Cantelli to show that $S_{n_k}$ exceeds the desired threshold infinitely often with probability zero or one. We then show that $S_n$ has the same behavior as the subsequence.
Proof of (??) Fixing $\varepsilon > 0$, choose $\theta$ such that $1 < \theta^2 < 1 + \varepsilon$. Define
\[ n_k = \left\lfloor \theta^k \right\rfloor, \quad x_k = \theta \sqrt{2 \log \log n_k}. \]
Note that $x_k = (1 + o(1))\theta \sqrt{2 \log k}$. Applying Lemmas ??, ??, we obtain
\[ P \left( \frac{M_{n_k}}{\sqrt{n_k}} \geq x_k \right) \leq 2 P \left( \frac{S_{n_k}}{\sqrt{n_k}} \geq x_k - \sqrt{2} \right) \]
\[ = 2 \exp \left( -\frac{1}{2}(x_k - \sqrt{2})^2 (1 + \xi_k) \right) \]
\[ = 2 \exp \left( -\frac{1}{2} \cdot 2\theta^2 \log k (1 + o(1)) \right) \]
\[ \leq \frac{2}{k^{\theta^2}}, \]
the last bound holding for all large $k$. As $\theta^2 > 1$, the RHS is summable, so by Borel-Cantelli I we have
\[ P \left( \frac{M_{n_k}}{\sqrt{n_k}} \geq x_k \text{ i.o.} \right) = 0. \]

We now argue that $S_n \geq (1 + \varepsilon)\sqrt{2n \log \log n}$ infinitely often will happen with probability zero. Suppose it happens infinitely often, let $n$ be an index where it happens. Let $k$ be such that $n_{k-1} < n \leq n_k$. We then have
\[ \frac{M_{n_k}}{x_k \sqrt{n_k}} = \frac{M_{n_k}}{\theta \sqrt{2n_k \log \log n_k}} \geq \frac{S_n}{\theta \sqrt{2n \log \log n}} \cdot \frac{2n_{k-1} \log \log n_{k-1}}{2n_k \log \log n_k} \]
\[ \geq \frac{1 + \varepsilon}{\theta} \cdot \sqrt{\frac{2\theta^{k-1} \log (k-1)}{2\theta^k \log k}} (1 + o(1)) \]
\[ \geq \frac{1 + \varepsilon}{\theta^{3/2}} (1 + o(1)). \]
As $1 + \varepsilon > \theta^2 > \theta^{3/2}$, for sufficiently large $k$, the above quantity will be greater than one. Hence, $M_{n_k}/\sqrt{n_k} \geq x_k$ will happen infinitely often. As this has probability zero, we must have $P(S_n \geq (1 + \varepsilon)\sqrt{2n \log \log n} \text{ i.o.}) = 0$, thereby showing (??).

Proof of (??) Let $\theta$ be an integer such that $3/\sqrt{\theta} < \varepsilon$ and $n_k = \theta^k$. Define
\[ a_k = x_k/\sqrt{n_k - n_{k-1}} \quad \text{with} \quad x_k = (1 - \theta^{-1})\sqrt{2n_k \log \log n_k}. \]
As $S_n$ are sums of independent R.V.-s, we can apply Lemma ?? to $S_{n_k} - S_{n_{k-1}}$ and get
\[ P \left( S_{n_k} - S_{n_{k-1}} \geq x_k \right) = \exp \left( -\frac{x_k^2}{2(n_k - n_{k-1})} (1 + \xi_k) \right) \]
\[ = \exp \left( -\frac{(1 - \theta^{-1})^2 2\theta^k \log k}{2(1 - \theta^{-1})\theta^k} (1 + o(1)) \right) \]
\[ = \exp \left( -(1 - \theta^{-1}) \log k(1 + o(1)) \right) \]
\[ \leq \frac{2}{k^{1-\theta^{-1}}}, \]
We now argue that the above implies \( S_{n_k} > (1 - \varepsilon)\sqrt{2n_k \log \log n_k} \) happens infinitely often with probability one, thereby showing the result. Indeed, applying the established result (??) to \(-S_{n_k}\) with \(\varepsilon = 1\), we get \(-S_{n_k-1} \leq 2\sqrt{2n_k \log \log n_k} - k\) for all large \(k\). Combined with the above result, we get that with probability one,

\[
S_{n_k} \geq x_k - 2\sqrt{2n_k \log \log n_k - 1} \geq x_k - \frac{2}{\sqrt{\theta}}\sqrt{2n_k \log \log n_k} = \left(1 - \frac{1}{\theta} - \frac{2}{\sqrt{\theta}}\right)\sqrt{2n_k \log \log n_k} \\
\geq \left(1 - \frac{3}{\sqrt{\theta}}\right)\sqrt{2n_k \log \log n_k} \geq (1 - \varepsilon)\sqrt{2n_k \log \log n_k}.
\]

For completeness, we also provide the proof of Lemma 2.

**Proof of Lemma 2.** Suppose \(M_n/\sqrt{n} \geq \alpha\), then either \(S_n/\sqrt{n} \geq \alpha - \sqrt{2}\), or \(S_n/\sqrt{n} < \alpha - \sqrt{2}\) and one of the following happens: \(M_{j-1} < \alpha \sqrt{n}\) but \(M_j \geq \alpha \sqrt{n}\). Defining \(A_j = \{M_{j-1} < \alpha \sqrt{n} \leq M_j\}\), then

\[
\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \sum_{j=1}^{n-1} \mathbb{P}\left(A_j \cap \left\{\frac{S_n}{\sqrt{n}} \leq \alpha - \sqrt{2}\right\}\right).
\]

On each of the event \(A_j \cap \{\cdots\}\), we have \(S_j \geq \alpha \sqrt{n}\) and \(S_n \leq (\alpha - \sqrt{2})\sqrt{n}\), which implies \((S_n - S_j)/\sqrt{n} \leq -\sqrt{2}\). This event is independent of \(A_j\), and \(S_n - S_j\) has variance \(n - j\), so we get

\[
\mathbb{P}\left(A_j \cap \left\{\frac{S_n}{\sqrt{n}} \leq \alpha - \sqrt{2}\right\}\right) \leq \mathbb{P}\left(A_j \cap \left\{\frac{S_n - S_j}{\sqrt{n}} \leq -\sqrt{2}\right\}\right) \\
= \mathbb{P}(A_j)\mathbb{P}\left(\frac{S_n - S_j}{\sqrt{n}} \leq -\sqrt{2}\right) \leq \frac{n - j}{2n} \mathbb{P}(A_j).
\]

Plugging into the preceding bound gives

\[
\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \sum_{j=1}^{n-1} \frac{n - j}{2n} \mathbb{P}(A_j) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \frac{1}{2} \sum_{j=1}^{n-1} \mathbb{P}(A_j).
\]

As \(A_j\) are disjoint and \(\bigcup A_j\) implies \(\{M_n/\sqrt{n} \geq \alpha\}\), we get

\[
\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \frac{1}{2} \mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right),
\]

from which the result follows. \(\square\)