Statistics 310A Session 8

December 10, 2019

In this session, we will go through some practice problems. These problems fall in the scope of Stats 310A and involve a lot of what we learned comprehensively.

**Problem 1** Let $X$ be a set, $B$ be a countably generated $\sigma$-algebra of subsets of $X$. Let $P(X, B)$ be the set of all probability measures on $(X, B)$. Make $P(X, B)$ into a measurable space by declaring that the map $P \mapsto P(A)$ is Borel measurable for each $A \in B$. Call the associated $\sigma$-algebra $B^*$.

(a) Show that $B^*$ is countably generated.

(b) For $\mu \in P(X, B)$, show that $\{\mu\} \in B^*$.

(c) For $\mu, \nu \in P(X, B)$, let

$$\|\mu - \nu\| = \sup_{A \in B} |\mu(A) - \nu(A)|.$$ 

Show that the map $(\mu, \nu) \mapsto \|\mu - \nu\|$ is $B^* \times B^*$ measurable.

**Problem 2** Let $\{X_n\}_n$ be iid symmetric random variables such that

$$\lim_{y \to \infty} \frac{y^2 \Pr(|X_1| > y)}{\mathbb{E}(X_1^2; |X_1| < y)} = 0. \tag{1}$$

Show that there exists a sequence $\{b_n\}_n$ of positive constants such that

$$\frac{1}{b_n} \sum_{k=1}^n X_k \overset{d}{\to} \mathcal{N}(0,1). \tag{2}$$

**Problem 3** Recall that given two measures $\mu, \nu$ on $(\mathbb{R}, B_\mathbb{R})$, a coupling of $\mu$ and $\nu$ is any probability measure $\gamma$ on $(\mathbb{R}^2, B_{\mathbb{R}^2})$ such that, for any Borel set $A$, we have $\gamma(A \times \mathbb{R}) = \mu(A)$, $\gamma(\mathbb{R} \times A) = \nu(A)$. (In words, the one-dimensional marginals of $\gamma$ are –respectively– $\mu$ and $\nu$.) We denote by $\Gamma(\mu, \nu)$ the set of couplings of $\mu$ and $\nu$. For $p \geq 1$, let $P_p$ be the space of probability measures $\mu$ such that $\int |x|^p \mu(dx) < \infty$. For $\mu, \nu \in P_p$, their $p$-th Wasserstein distance is

$$W_p(\mu, \nu) = \left\{ \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \gamma(dx, dy) \right\}^{1/p} \tag{3}$$

1. For $\mu = \mathcal{N}(0,1)$ and $\nu = \mathcal{N}(a,1)$, prove that $W_2(\mu, \nu) = |a|$.

2. For $\mu = \mathcal{N}(0,1)$ and $\nu = \mathcal{N}(0,v)$, $v > 1$, prove that $W_2(\mu, \nu) = \sqrt{v} - 1$.

3. Prove that $\Gamma(\mu, \nu)$ is uniformly tight.
4. Fix $p \geq 1$. Prove that there exists a sequence of probability measures $\{\gamma_n\}_{n \in \mathbb{R}} \subseteq \Gamma(\mu, \nu)$ and $\gamma \in \Gamma(\mu, \nu)$ such that $\gamma_n \overset{w}{\rightarrow} \gamma$, and

$$
\lim_{n \to \infty} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \gamma_n(dx, dy) = W_p(\mu, \nu)^p.
$$

(4)

5. Prove that (for $\{\gamma_n\}_{n \in \mathbb{N}}$, $\gamma$ constructed as in the previous point)

$$
\lim \inf_{n \to \infty} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \gamma_n(dx, dy) \geq \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \gamma(dx, dy),
$$

(5)

and deduce that

$$
W_p(\mu, \nu) = \left\{ \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \gamma(dx, dy) \right\}^{1/p}
$$

(6)