Lecture 1

Stochastic processes : collections of r.v.'s $\{X_t\}_{t\in\mathbb{T}}$
indexed by $\mathbb{T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$
$\mathbb{T}$ uncountable $\quad \mathbb{T} = [a,b] \subseteq \mathbb{R}$

- Definitions / constructions
- Filtrations, MG
- Brownian motion

How do we specify $(X_t)_{t\in T}$?

Countable case $\quad T = \mathbb{N} \quad (X_t)_{t\in\mathbb{N}}$ takes values in $\mathbb{R}^{\mathbb{N}} =: \Omega$

$$\mathcal{B}_c := \sigma(\{ R^{(n)}_{t_1 \dots t_n}(A_1, \dots A_n) : t_1 \dots t_n \in \mathbb{N}, A_1 \dots A_n \in \mathcal{B}\})$$

$$R^{(n)}_{t_1 \dots t_n}(A_1 \dots A_n) := \{ x \in \mathbb{R}^{\mathbb{N}} : x(t_1) \in A_1 \dots x(t_n) \in A_n \})$$

Thm (Kolmogorov) Suppose $\forall n$ μ_n prob measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ st

$$\mu_{n+1}(A_1 \times \dots \times A_n \times \mathbb{R}) = \mu_n(A_1 \times \dots \times A_n)$$

Then $\exists !$ $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_c)$ st $\forall n$

$$\mathbb{P}(R^{(n)}_{1,2,\dots n}(A_1, A_2 \dots A_n)) = \mu_n(A_1 \times \dots \times A_n) \quad \square$$

Rmk Will talk about processes taking val in $\mathbb{R}$
Generalizes to $\mathcal{B}$-equiv. spaces.

Eg : Polish spaces (complete, metric, separab)
($\mathbb{R}^d$)

<u>Finite dimens. distr</u> (fdd)

$\mu_{t_1 \dots t_n}$ prob distr on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$

$t_1 \dots t_n \in \mathbb{T}$. distinct

<u>Consistent</u> if $\forall t_1 .. t_n, t_{n+1}, \pi \in S_n \; \forall A_1 .. A_n \in \mathcal{B}$.

(i) $\mu_{t_1 \dots t_n}(A_1 \times .. \times A_n) = \mu_{t_{\pi(1)} \dots t_{\pi(n)}}(A_{\pi(1)} \times \dots \times A_{\pi(n)})$

(ii) $\mu_{t_1 \dots t_n, t_{n+1}}(A_1 \times .. \times A_n \times \mathbb{R}) = \mu_{t_1 .. t_n}(A_1 \times .. \times A_n)$

<u>Rmk</u> $\mathbb{T} \subseteq \mathbb{R}$ then by (i) sufficient to give
$\mu_{..}$ for $t_1 < t_2 < .. < t_n$.
(ii) satisfied iff

$$\mu_{t_1 \dots t_n}(A_1 \times .. \times \underbrace{\mathbb{R}}_{i\text{-th pos.}} \times \dots A_n) = \mu_{t_1 \dots t_{i-1} t_{i+1} \dots t_n}(A_1 \times \dots A_n) \; . \; \square$$

K. thm : If $\mathbb{T} = \mathbb{N} \Rightarrow$ consistent fdd determine unique $\mathbb{P}$ on $\mathcal{B}_c$

<u>Can we generalize this</u> ?

$\mathbb{R}^{\mathbb{T}} := \{ x : \mathbb{T} \to \mathbb{R}, \; t \mapsto x(t) \}$

$R^{(n)}_{t_1 .. t_n}(A_1, \dots A_n)$

$\mathcal{B}^{\mathbb{T}} := \sigma(\{ R^{(n)}_{t_1 .. t_n}(A_1, \dots A_n) \})$

Def $A \subseteq \mathbb{R}^T$ has coutable repr if $\exists\, S = (t_1, t_2, \dots)$
$D \in \mathcal{B}_c \subseteq \mathbb{R}^{\mathbb{N}}$ st.

$$A = \{ x \in \mathbb{R}^T : \underbrace{(x(t_1), x(t_2), \dots)}_{\pi_S(x)} \in D \}$$

S "base" ; (S, D) representation

$A = \pi_S^{-1}(D)$

Lemma $\mathcal{B}^T = \mathcal{C} := \{ A \subseteq \mathbb{R}^T : A \text{ has count. repr} \}$

Proof ① $\boxed{\mathcal{C} \subseteq \mathcal{B}^T}$ fix $S = (t_1, t_2, \dots)$

$\mathcal{C}(S) = \{ \pi_S^{-1}(D) : D \in \mathcal{B}_c \}$ ✓

claim $\mathcal{C}(S) \subseteq \mathcal{B}^T$

Indeed $\pi_S : (\mathbb{R}^T, \mathcal{B}^T) \to (\mathbb{R}^{\mathbb{N}}, \mathcal{B}_c)$ measurable
because

$$\pi_S^{-1}(A_1 \times \dots \times A_n \times \mathbb{R} \times \dots) \in \mathcal{B}^T$$
$$(= R^{(n)}_{t_1 \dots t_n}(A_1, \dots A_n))$$

② $\boxed{\mathcal{B}^T \subseteq \mathcal{C}}$

- $R^{(n)}_{t_1 \dots t_n}(A_1, \dots A_n) \in \mathcal{C}$
- Claim $\mathcal{C}$ is a σ-algebra

$A_i \in \mathcal{C}$ wts $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$

A_i, (S_i, D_i) $A_i = \pi_{S_i}^{-1}(D_i)$

$S = \bigcup_{i=1}^{\infty} S_i \qquad \pi_{S \to S_i}$ proj from S to S_i

$D = \bigcup_{i=1}^{\infty} \pi^{-1}_{S \to S_i}(D_i) \quad \in \mathcal{B}_c$

(S, D) is a represent of $A = \bigcup_{i=1}^{\infty} A_i$ $\square$

$(X_t)_{t \in \mathbb{T}} \qquad \mathcal{F}^X :=$ "smallest" σ-algebra such that $X_t \in m\,\mathcal{F}^X \quad \forall t \in \mathbb{T}$.

<u>Lemma</u> $\mathcal{F}^X = \{ \underbrace{X_\cdot^{-1}(A)}_{\{\omega \in \Omega : [X_t(\omega)]_{t \in \mathbb{T}} \in A\}} : A \in \mathcal{B}^{\mathbb{T}} \}$ $\square$

Btw $\quad t \mapsto X_t(\omega) \quad$ sample path $\square$

<u>Thm</u> Given fdd's $(\mu_{t_1 \ldots t_n})$, consistent
$\exists \ (\Omega, \mathcal{F}, \mathbb{P})$ and $X_t : \Omega \to \mathbb{R} \quad \forall t \in \mathbb{T}$
whose fdd are given by μ.

$$\mathbb{P}(R^{(n)}_{t_1 - t_n}(A_1 \ldots A_n)) = \mu_{t_1 \ldots t_n}(A_1 \times \ldots \times A_n).$$

Further $\mathbb{P}|_{\mathcal{F}^X}$ unique. $\square$

(Given $\mathbb{P}, \mathbb{Q}$ with same fdd's

$$\mathbb{P}(\{\omega : X_\cdot(\omega) \in B\}) = \mathbb{Q}(\{\omega : X_\cdot(\omega) \in B\})$$

$\forall B \in \mathcal{B}^{\mathbb{T}}$.

I know $\mathbb{P}(\{\omega : X_\cdot(\omega) \in R_{t_1 \ldots t_n}(A_1, \ldots A_n)\}) = \mathbb{Q}(\ )$

<u>Proof</u> $\Omega = \mathbb{R}^{\mathbb{T}}$, $\mathcal{F} = \mathcal{B}^{\mathbb{T}}$; $X_t(\omega) = \omega(t)$

Fix $S = (t_1, t_2, \dots)$ can define μ_S on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_c)$ via K-thm.

Hence $\forall A \in \mathcal{C}(S)$, $A = \pi_S^{-1}(D)$ $D \in \mathcal{B}_c$

$\mathbb{P}(A) = \mu_S(D)$.

(1) Well defined $A = \pi_{S_1}^{-1}(D_1) = \pi_{S_2}^{-1}(D_2)$

need to show $\mu_{S_1}(D_1) = \mu_{S_2}(D_2)$

(idea : look at $S = S_1 \cup S_2$)

(2) $A_k = \pi_{S_k}^{-1}(D_k)$ disjoint. Wts.

$$\mathbb{P}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$$

Idea : represent A_k on $S = \bigcup_{k=1}^{\infty} S_k$

$$A_k = \pi_{S_k}^{-1}(D_k) = \pi_S^{-1}(\tilde{D}_k)$$

$$\bigcup_{k=1}^{\infty} A_k = \pi_S^{-1}\Big(\bigcup_{k=1}^{\infty} \tilde{D}_k\Big) \qquad \tilde{D}_k \text{ disjoint}$$

$$\mathbb{P}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \mu_S\Big(\bigcup_{k=1}^{\infty} \tilde{D}_k\Big) = \sum_{k=1}^{\infty} \mu_S(\tilde{D}_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$$

$\square$

Unfortunately many interesting events not in $\mathcal{B}^{\mathbb{T}}$!

<u>Example</u> $\mathbb{T} = [0,1]$ $A \subseteq \mathbb{R}^{[0,1]}$

$A = \{x(t) \geq 0 \ \forall t \in [0,1]\} \notin \mathcal{B}^{\mathbb{T}}$

If it was there would be $(t_1, t_2 \ldots) = S$

$D \in \mathcal{B}_c$ st

$$A = \{x : (x(t_1), x(t_2), \ldots) \in D\}$$

$x|_S = y|_S \Rightarrow$ either $x, y \in A$ or $x, y \notin A$

Take $x(t) = 0 \ \forall t$

$$y(t) = 0 \ \forall t \neq t_* \qquad y(t_*) = -1$$

$t_* \notin S \qquad x \in A \qquad y \notin A \qquad \pi_S(x) = \pi_S(y)$ $\square$

In general fdd do not determine the prob of these events.

<u>Example</u> $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_{[0,1]}, \text{Unif})$

$(X_t)_{t \in [0,1]}$ $\qquad X_t(\omega) = \begin{cases} 1 & \text{if } t \neq \omega \\ -1 & \text{if } t = \omega \end{cases}$

$$\tilde{X}_t(\omega) = 1. \quad \text{always.}$$

Same fdd.

$$\mu_{t_1 \ldots t_n}(A_1 \times \ldots \times A_n) = \mathbb{1}(1 \in A_1, \ldots 1 \in A_n)$$

$$\mathbb{P}(\tilde{X}_t \geq 0 \ \forall t) = 1. \qquad \mathbb{P}(X_t \geq 0 \ \forall t) = 0.$$