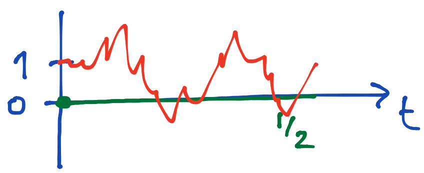
Lecture 10   Do all homogeneous MP have strong Markov property?

Example $\{W_t\}_{t\geq 0}$ std Wiener indep of X_0

$$\mathbb{P}(X_0=0)=\frac{1}{2}\ ,\ \mathbb{P}(X_0=1)=\frac{1}{2}$$

$$X_t = X_0(1+W_t)$$



$$\mathbb{P}(X_{t+u}\in B \mid \mathcal{F}_t^X) = \mathbb{P}(X_0(1+W_{t+u})\in B \mid \mathcal{F}_t^X)$$

$$= \mathbb{E}\left[I_{X_0(1+W_{t+u})\in B}\, X_0 \mid \mathcal{F}_t^X\right] + \mathbb{E}\left[I_{X_0(1+W_{t+u})\in B}(1-X_0)\mid \mathcal{F}_t^X\right]$$

$$= X_0\, \mathbb{P}(W_{t+u}-W_t \in X_t + B \mid \mathcal{F}_t^X) + (1-X_0)\, I_{0\in B}$$

$$= \underline{X_0}\ p_u^W(X_t, B) + (1-\underline{X_0})\ p_0^W(0,B)$$

$X_0 = I_{X_t\neq 0}$ a.s. $(\ \mathbb{P}(X_t=0 \mid X_0\neq 0)=0)$

$$\mathbb{P}(X_{t+u}\in B\mid \mathcal{F}_t^X) = I_{X_t\neq 0}\, p_u^W(X_t,B) + I_{X_t=0}\, p_0^W(0,B).$$

$$p_u(x,B) = I_{x\neq 0}\, p_u^W(x,B) + I_{x=0}\, p_0^W(x,B). \quad \checkmark$$

Is this a trans. probability?

$$\int_{\mathbb{R}} p_s(x_1,B)\, p_t(x_0,dx_1) = I_{x_0\neq 0}\int_{\mathbb{R}} p_s(x_1,B)\, p_t^W(x_0,dx_1)$$
$$+ I_{x_0=0}\, p_s(0,B)$$
$$= I_{x_0\neq 0}\int_{\mathbb{R}} p_s^W(x_1,B)\, p_t^W(x_0,dx_1) + I_{x_0=0}\, p_0^W(0,B)$$
$$= I_{x_0\neq 0}\, p_{t+s}^W(x_0,B) + I_{x_0=0}\, p_0^W(x_0,B) = p_{s+t}(x_0,B)$$

$$\tau := \inf\{t>0 : X_t=0\}$$

If it had sMP $\qquad B \not\ni 0$

$$\mathbb{P}(X_{\tau+u}\in B \mid \mathcal{F}_\tau^X) = p_u(X_\tau, B) = p_u(0,B) = 0$$

$$\mathbb{P}(X_{\tau+u}\in B \mid \mathcal{F}_\tau^X) = \mathbb{E}[X_0\, I_{X_{\tau+u}-X_\tau\in B-X_\tau} \mid \mathcal{F}_\tau^X]$$
$$= X_0\ \mathbb{E}[I_{W_{\tau+u}-W_\tau\in B} \mid \mathcal{F}_\tau^X] = X_0\, p_u^W(0,B)$$

Use Wiener has SMP

$B=(0,\infty)$
$$= \frac{1}{2}\cdot X_0 \neq 0$$

<u>Def</u> A Markov semigroup p_t on $(\mathcal{X},\mathcal{G})$ (Polish) is Feller if $\forall t$.
$$f\in C_b(\mathcal{X}) \Longrightarrow p_t f \in C_b(\mathcal{X})$$ □

<u>Lemma</u> $(X_t)_{t\geq 0}$ homogeneous MP $\tau:\Omega\to[0,\infty]$ stopping time with values in $S\subseteq[0,\infty]$ count.

Then $\forall\, h \in b(\mathcal{B}_{[0,\infty)} \times \mathcal{G}^{[0,\infty)})$; $S = \{s_k\} \cup \{\infty\}$

$$(*) \qquad \mathbb{E}[h(\tau, \theta_\tau \circ X) \mid \mathcal{F}_\tau] I_{\tau<\infty} = g(\tau, X_\tau) I_{\tau<\infty}$$

$$g(s,x) := \mathbb{E}_x[h(s,X)]$$

$\boxtimes$

Proof $\forall s_k \quad h(s_k, \cdot) \in b(\mathcal{G}^{[0,\infty)})$

$A \in \mathcal{F}_\tau$

$$\mathbb{E}[h(\tau, \theta_\tau \circ X) I_A I_{\tau<\infty}] = \sum_{k=1}^{\infty} \mathbb{E}[h(\tau, \theta_\tau \circ X) I_{A \cap \{\tau = s_k\}}]$$

$$= \sum_{k=1}^{\infty} \underline{\mathbb{E}[h(s_k, \theta_{s_k} \circ X) I_{A \cap \{\tau = s_k\}}]}$$

Note $\overline{A \cap \{\tau = s_k\}} \in \mathcal{F}_{s_k}$

[indeed $\{\tau = s_k\} = \{\tau \le s_k\} \setminus \bigcup_{j: s_j < s_k} \{\tau \le s_j\}$]

$$\ldots = \sum_{k=1}^{\infty} \mathbb{E}[\mathbb{E}[h(s_k, \theta_{s_k} \circ X) \mid \mathcal{F}^X_{s_k}] I_{A \cap \{\tau = s_k\}}]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}[g(s_k, X_{s_k}) I_{A \cap \{\tau = s_k\}}]$$

$$= \mathbb{E}[g(\tau, X_\tau) I_A I_{\tau<\infty}].$$

$\boxtimes$

$$\mathbb{E}[h(t, \theta_t \circ X) \mid \mathcal{F}^X_t] = g(t, X_t) \qquad \forall t.$$

Theorem $(X_t, \mathcal{F}_t)$ right cont. homogeneous Markov with Feller semigroup $(p_t)_{t\geq 0}$. Then $(X_t, \mathcal{F}_t)$ is strong Markov

Proof $(\mathcal{X}, \mathcal{G}) = (\mathbb{R}, \mathcal{B})$. Recall that it is sufficient to show $\forall \tau$ bounded Markov $\forall B \in \mathcal{B}$. $\forall u \in \mathbb{R}_{\geq 0}$

$$\mathbb{P}(X_{\tau+u} \in B \mid \mathcal{F}_{\tau+}) = p_u(X_\tau, B) \qquad (*)$$

Will prove $\forall f \in C_b(\mathbb{R})$ $\forall A \in \mathcal{F}_{\tau+}$ $\forall u$

$$\mathbb{E}[f(X_{\tau+u}) I_A] = \mathbb{E}[(p_u f)(X_\tau) I_A] \qquad (**)$$

[This implies $(*)$. Indeed $B = (a,b)$ $f_k \uparrow I_{(a,b)}$ by monotone conv

$\hookrightarrow$ $\mathbb{E}[I_{X_{\tau+u} \in B} I_A] = \mathbb{E}[(p_u I_B)(X_\tau) I_A]$

hence this holds $\forall B \in \mathcal{B}$ by π-λ

hence $(*)$ since $A \in \mathcal{F}_{\tau+}$ arbitrary] need to prove

(**) $\mathbb{E}[f(X_{\tau+u}) I_A] = \mathbb{E}[(p_u f)(X_\tau) I_A]$

$\tau_\ell := \inf\{ t = k2^{-\ell} : t > \tau \}$.

$\tau_\ell \downarrow \tau$ $\qquad \tau < \tau_\ell \leq \tau + 2^{-\ell}$ $\qquad \tau_\ell = 2^{-\ell}\lceil \tau 2^\ell \rceil - 1$

- τ_ℓ is a stopping time ($\{\tau_\ell \leq t\} \in \mathcal{F}_t$)

- $\mathcal{F}_{\tau+} \subseteq \mathcal{F}_{\tau_\ell}$ (because $\tau_\ell > \tau$)

$\Rightarrow$ $A \in \mathcal{F}_{\tau_\ell}$.

by lemma. $\forall h \in b(\mathcal{B}_{[0\infty)} \times \mathcal{Y}^{[0\infty)})$

$\mathbb{E}[h(\tau_\ell, \theta_{\tau_\ell} \circ X) | \mathcal{F}_{\tau_\ell}] = g(\tau_\ell, X_{\tau_\ell})$

$g(s,x) = \mathbb{E}_x h(s, X)$.

$h(s,X) = f(X_u)$ $\qquad g(s,x) = \mathbb{E}_x f(X_u) = (p_u f)(x)$

Hence

$$\mathbb{E}[f(X_{\tau_\ell+u}) | \mathcal{F}_{\tau_\ell}] = (p_u f)(X_{\tau_\ell})$$

$\Rightarrow$ $\mathbb{E}\{f(X_{\tau_\ell+u}) I_A\} = \mathbb{E}[(p_u f)(X_{\tau_\ell}) I_A]$

Take $\ell \to \infty$ $\quad \tau_\ell \downarrow \tau$, $\quad X_{\tau_\ell} \to X_\tau$ $\quad X_{\tau_\ell+u} \to X_{\tau+u}$

(right cont). $\quad f(X_{\tau_\ell+u}) \to f(X_{\tau+u})$ $\quad (f \in C_b)$

lhs $\to \mathbb{E}[f(X_{\tau+u}) I_A]$ by dom.

$(p_u f)(X_{\tau_\ell}) \to (p_u f)(X_\tau)$ Because $p_u f$ cont. (Feller)

rhs $\to \mathbb{E}[(p_u f)(X_\tau) I_A]$

Hence

$$\mathbb{E}[f(X_{\tau+u}) I_A] = \mathbb{E}[(p_u f)(X_\tau) I_A] \qquad \blacksquare$$

Example of non Feller

$$p_u(x,B) = I_{x\neq 0}\, p_u^W(x,B) + I_{x=0}\, p_0^W(x,B)$$

$$(p_u f)(x) = \int_{\mathbb{R}} f(y)\, p_u(x,dy)$$

$$= I_{x\neq 0} \underbrace{\mathbb{E}_G[f(x+\sqrt{u}G)]} + I_{x=0} f(0) \qquad G\sim N(0,1)$$

$$\lim_{x\downarrow 0} (p_u f)(x) = \mathbb{E}_G[f(\sqrt{u}G)] \neq (p_u f)(0) = f(0) \qquad \blacksquare$$

Example of Feller $\quad X_t = X_0 + W_t \quad$ W_t std Wiener, X_0 indep of W_t

$$p_t(x,dy) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}\, dy$$

$$(p_t f)(x) = \mathbb{E}_G f(x+\sqrt{t}G) \qquad G\sim N(0,1)$$

cont in x by bdd conv.