Lecture 11 Brownian motion

$(W_t)_{t\geq 0}$ Gaussian, cont. paths, $\mathbb{E}(W_t W_s) = t \wedge s$

$W_0 = 0$ std Wiener process

$(B_t)_{t\geq 0}$ $B_t = X_0 + W_t$ X_0 indep of $(W_t)_{t\geq 0}$

Brownian Markov. $(W_t, \mathcal{F}_t)$, $\mathcal{F}_t = \mathcal{F}^W$

Transformations

All of $\tilde{W}_t$'s are std Wiener

1) $\tilde{W}_t = -W_t$

2) $\tilde{W}_t = \tilde{W}_{t_0+t} - \tilde{W}_{t_0}$ Fixed $t_0 \geq 0$

3) $\tilde{W}_t = W_{t_0-t} - W_{t_0}$ Fixed $t_0 \geq 0$, $t \in [0, t_0]$

4) $\tilde{W}_t = \alpha^{-1/2} W_{\alpha t}$ $\alpha > 0$

5) $\tilde{W}_t = t\, W_{1/t}$

[$\tilde{W}_t$ cont sample paths, Check fdd's.

Need only to check covariance

$\mathbb{E}(\tilde{W}_t \tilde{W}_s) = t \wedge s$.]

Strong Markov process

$h \in b(\mathcal{B}_{[0,\infty)} \times \mathcal{B}^{[0,\infty)})$, Markov time τ

(*) $\mathbb{E}[h(\tau, \theta_\tau \circ B) | \mathcal{F}_{\tau+}] 1_{\tau<\infty} = g(\tau, B_\tau) 1_{\tau<\infty}$

$g(s,x) = \mathbb{E}_x h(s,B)$

(equivalently $B_\cdot$ is a r.v with values in $(C([0,\infty)), \mathcal{B}_{C([0,\infty))})$, h bounded measurable on $C([0,\infty))$)

<u>Corollary</u> $(B_t)_{t\geq 0}$ is an $\mathcal{F}_{t+}$ Markov process and $\forall\, h : C([0,\infty)) \to \mathbb{R}$ meas. bdd.

$$\mathbb{E}[h(B) | \mathcal{F}_{t+}] = \mathbb{E}[h(B) | \mathcal{F}_t]$$

<u>Proof</u> Applying (*) to $\tau = t$ a.s.

$$\mathbb{E}[h(\theta_t \circ B) | \mathcal{F}_{t+}] = g(B_t)$$

□

<u>Theorem</u> (Blumenthal 0-1 law) $\mathbb{P}_x$ law of BM st. at x

(1) $A \in \mathcal{F}_{0+}^W \Rightarrow \mathbb{P}_x(A) \in \{0,1\}$.

(2) $A \in \mathcal{T}^W := \bigcap_{t\geq 0} \mathcal{T}_t^W$ $\quad \mathcal{T}_t^W := \sigma(\{W_s : s \geq t\})$

$\Rightarrow$ Either $\mathbb{P}_x(A) = 0\ \forall x$ or $\mathbb{P}_x(A) = 1\ \forall x$.

Rmk $\quad A = \{W_0 \in [a\, b]\} \in \mathcal{F}^W_{0+}$

$$\mathbb{P}_x(A) = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{if } x \notin [a, b] \end{cases} \qquad \square$$

Proof (1) By corollary $\forall A \in \mathcal{F}^W_\infty$

$$\mathbb{E}[I_A \mid \mathcal{F}^W_{0+}] = \mathbb{E}[I_A \mid \mathcal{F}^W_0] = \mathbb{P}_x(A)$$

$W_0 \underset{a.s.}{=} x \Rightarrow \mathcal{F}^W_0$ trivial.

$A \in \mathcal{F}^W_{0+}$

$$\mathbb{E}[I_A \mid \mathcal{F}^W_{0+}] \overset{a.s.}{=} \underset{\{0,1\}}{\underset{\cap}{I_A}} = \mathbb{P}_x(A) \Rightarrow \mathbb{P}_x(A) \in \{0,1\} \qquad \square$$

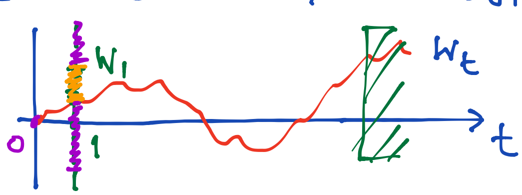
(2) First $x=0$.

Define $\quad X_t = t W_{1/t} \qquad X_0 = 0$

$(X_t, \mathcal{F}^X_t)$ is a BM. $\quad \mathcal{F}^X_t = \mathcal{T}^W_{1/t}$

$$\mathcal{T}^W = \bigcap_{k=1}^\infty \mathcal{T}^W_k = \bigcap_{k=1}^\infty \mathcal{F}^X_{1/k} = \mathcal{F}^X_{0+}$$

Hence $A \in \mathcal{T}^W \Rightarrow A \in \mathcal{F}^X_{0+} \Rightarrow \mathbb{P}_0(A) \in \{0,1\}$



$A \in \mathcal{T}^W_1 \Rightarrow A = \theta_1^{-1}(D) \qquad D \in \mathcal{F}^W_\infty$

$$\mathbb{P}_0(A) = \mathbb{E}_0\{ I_D \circ \theta_1 \} = \mathbb{E}_0\{ \mathbb{E}[I_D \circ \theta_1 \mid W_1] \}$$

$$= \mathbb{E}_0[\mathbb{P}_{W_1}(D)]$$

$$\mathbb{P}_0(A) = \int_{\mathbb{R}} \mathbb{P}_y(D) \quad \phi_1(0, dy) = \int_{\mathbb{R}} \mathbb{P}_y(D)\, \phi(y)\, dy$$

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$\mathbb{P}_0(A) = 0 \iff \mathbb{P}_y(D) = 0 \quad \text{a.e } y$$

$$\mathbb{P}_0(A) = 1 \iff \mathbb{P}_y(D) = 1 \quad \text{a.e } y.$$

$$\mathbb{P}_x(A) = \int_{\mathbb{R}} \mathbb{P}_y(D)\, \phi(y-x)\, dy$$

$$\mathbb{P}_0(A) = 0 \Rightarrow \mathbb{P}_x(A) = 0 \quad \forall x.$$

$$\mathbb{P}_0(A) = 1 \Rightarrow \mathbb{P}_x(A) = 1 \quad \forall x \qquad \square$$

<u>Corollary</u> W_t std Wiener

$$\tau_{0+} := \inf\{t > 0 : W_t > 0\}$$

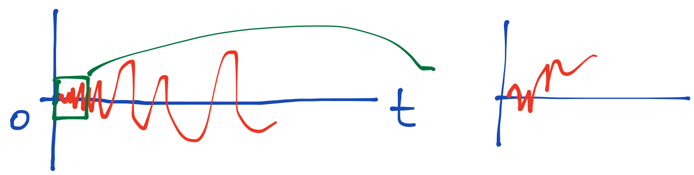
$$\tau_{0-} := \inf\{t > 0 : W_t < 0\}$$

$$\tau_0 := \inf\{t > 0 : W_t = 0\}$$

Then $\mathbb{P}_0(\tau_{0+} = 0) = \mathbb{P}_0(\tau_{0-} = 0) = \mathbb{P}_0(\tau_0 = 0)$

Further $\forall x \in \mathbb{R}$ $\mathbb{P}_x$ a.s.

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t}} W_t = \infty, \quad \liminf_{t\to\infty} \frac{1}{\sqrt{t}} W_t = -\infty.$$



Proof τ_{0+} Markov time (open set, right cont)

$$\{\tau_{0+}=0\} \in \mathcal{F}^W_{0+} \Rightarrow \mathbb{P}_0(\tau_{0+}=0) \in \{0,1\}$$

$$\mathbb{P}(\tau_{0+}=0) = \lim_{t\downarrow 0} \mathbb{P}(\tau_{0+}\leq t)$$

$$= \lim_{t\to 0} \mathbb{P}(\exists s\in[0,t] \text{ st } W_s>0)$$

$$\geq \lim_{t\to 0} \mathbb{P}(W_t>0) = 1/2$$

$$\Rightarrow \mathbb{P}(\tau_{0+}=0)=1$$

$\square$

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{1}{\sqrt{t}} W_t \geq r\right) \geq \mathbb{P}\left(\frac{1}{\sqrt{n}} W_n \geq r \text{ i.o.}\right)$$

$$= \lim_{n_0\to\infty} \mathbb{P}\left(\bigcup_{n\geq n_0} \left\{\frac{1}{\sqrt{n}} W_n \geq r\right\}\right)$$

$$\geq \lim_{n_0\to\infty} \underbrace{\mathbb{P}\left(\frac{1}{\sqrt{n_0}} W_{n_0} \geq r\right)}_{\mathbb{P}(W_1\geq r)} = \mathbb{P}(W_1\geq r) > 0$$

By 0-1 law $\mathbb{P}\left(\limsup_{t\to\infty} \frac{1}{\sqrt{t}} W_t \geq r\right) = 1$.

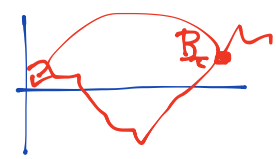
Take $r_k \uparrow \infty$.

$\square$

Corollary $(B_t, \mathcal{F}_t)$ Brownian MP. τ Markov time $\tau < \infty$ a.s. Then $(B_{t+\tau} - B_\tau)_{t \geq 0}$ is st Wiener indep of $\mathcal{F}_{\tau+}$ $\boxtimes$

Proof $X_t := B_{t+\tau} - B_\tau$

$(X_t)_{t \geq 0}$ is a sp. with cont sample paths.



$h \in b\mathcal{B}^{[0,\infty)}$, $A \in \mathcal{F}_{\tau+}$

$$\mathbb{E}[h(X) I_A] = \mathbb{E}[h(B_{\tau+\cdot} - B_\tau) I_A]$$

define $\tilde{h}(x(\cdot)) = h(x(\cdot) - x(0))$

$$(*) = \mathbb{E}[\tilde{h} \circ \theta_\tau (B) I_A]$$

$$= \mathbb{E}[g(B_\tau) I_A] \qquad (W_{t+t_0} - W_{t_0})_{t \geq 0} \stackrel{d}{=} (W_t)_{t \geq 0}$$

where $g(x) = \mathbb{E}_x \tilde{h}(B) = \mathbb{E}_x \tilde{h}(W) = \mathbb{E}_x h(W_\cdot - W_0)$

$$= \mathbb{E}_0 h(W)$$

$$(*) = \mathbb{E}[\mathbb{E}_0 h(W)] \cdot I_A] = \mathbb{E}_0 h(W) \cdot \mathbb{P}(A).$$

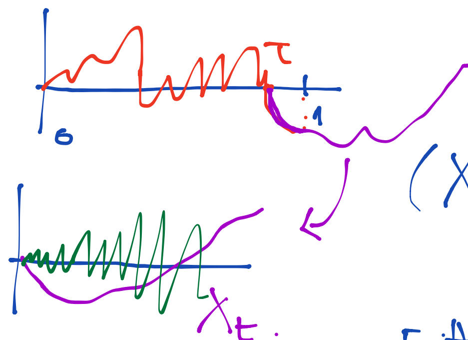
$$\mathbb{E}[h(X) I_A] = \mathbb{E}_0 h(W) \cdot \mathbb{P}(A)$$

$\boxtimes$

Crucial that τ is Markov!

<u>Example</u> $\tau = \sup\{ t\in[0,1] : W_t = 0 \}$

$$\mathbb{P}(\tau < 1) = 1$$



$(W_{t+\tau} - W_\tau) =: X_t$

$(X_t)_{t\geq 0}$ is very different from Wiener!

Either $W_t > 0 \ \forall t\in(\tau,1]$

or $W_t < 0 \ \forall t \in [\tau,1]$

$$1 = \mathbb{P}(\exists \epsilon > 0 \text{ st } X_t \neq 0 \ \forall t\in(0,\epsilon])$$

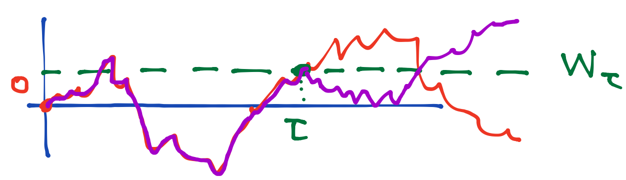
$$= \mathbb{P}(T_0^X > 0) \qquad \square$$

<u>Proposition</u> (Reflection principle) $(W_t, \mathcal{F}_t)_{t\geq 0}$

τ an $\mathcal{F}_t$ Markov time. ($\tau < \infty$ a.s.)

$$W_t^* = W_t \mathbb{1}_{t\leq\tau} + (2W_\tau - W_t)\mathbb{1}_{t>\tau}.$$

Then (W_t^*) is std Wiener. $\square$



W_t^* is Wiener

<u>Proof</u>

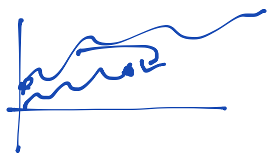
$X = \left((W_t)_{t \le \tau} ,\ (W_{t+\tau} - W_\tau)_{t > \tau} \right)$ indep.

$X' = \left((W_t)_{t \le \tau} ,\ -(W_{t+\tau} - W_\tau)_{t > \tau} \right)$ indep

$X \overset{d}{=} X'$ $\qquad$ $g =$ concatenation.

$g(x_1, x_2) = x$ $\qquad$ $x_1 \in C([0,t_1])$

$x(t) = \begin{cases} x_1(t) & t \le t_1 \\ x_1(t_1) + x_2(t - t_1) & t > t_1 \end{cases}$ $\qquad$ $x_2 \in C([0,\infty))$



Claim: g measurable

Hence $g(X) \overset{d}{=} g(X')$ $\square$

<u>Proposition</u> $(W_t)_{t \ge 0}$ std Wiener $b > 0$

$M_t = \sup\{ W_s : s \in [0,t] \}$.

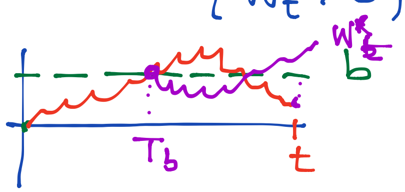
$\tau_{b+} = \inf\{ t > 0 : W_t > b \}$

$\mathbb{P}(M_t > b) = \mathbb{P}(\tau_{b+} < t) = 2\,\mathbb{P}(W_t > b)$ $\square$

<u>Proof</u>

$\{M_t > b\} = \{M_t > b;\ W_t > b\} \cup \{M_t > b;\ W_t \le b\}$

$= \{W_t > b\} \cup \{M_t > b\ ;\ W_t \le b\}$



$T_b = \inf\{t : W_t = b\}$

$\mathbb{P}(M_t > b) = \mathbb{P}(W_t > b) + \mathbb{P}(M_t > b,\ W_t \le b)$

$$= \mathbb{P}(W_t > b) + \mathbb{P}(W_t^* \geq b)$$

$$= 2\mathbb{P}(W_t > b). \qquad \square$$