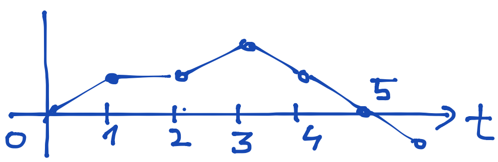
Lecture 12 — Brownian motion and R.W.

$(\xi_k)_{k\geq 1}$ iid $\quad \mathbb{E}\xi_k=0,\ \mathbb{E}\xi_k^2=1$

$$S(t)=\sum_{k=1}^{[t]}\xi_k+(t-\lfloor t\rfloor)\,\xi_{\lfloor t\rfloor+1}$$



$$\hat S_n(t)=\frac{1}{\sqrt n}S(nt)$$

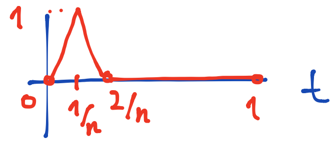
want to show that $\hat S_n$ converges to Wiener as $n\to\infty$.

<u>Def</u> $(X_n(t))_{t\in[0,\infty)}$ $(X_\infty(t))_{t\in[0,\infty)}$ SP with cont sample paths. We say $X_n\xrightarrow{d}X_\infty$ (in distr) if $\forall$ h bdd cont on $C([0,\infty))$

$$\lim_{n\to\infty}\mathbb{E}h(X_n)=\mathbb{E}h(X_\infty)\ \square$$

- h bdd, cont wrt topology of unif conv on compacts ($x_n\to x_\infty$ if $\forall T>0$ $\sup_{0\le t\le T}|x_n(t)-x_\infty(t)|\to 0$)

- Convergence in distr wrt topology of unif conv on compacts.



$x_n(t) \to x_\infty(t)$ $\forall t \in [0,1]$

$x_\infty(t) = 0$ $\forall t \in [0,1]$

$\Leftrightarrow$ $\forall h$ bdd such that $\mathbb{P}(X_\infty \in \mathcal{C}_h) = 1$

$\mathbb{E}h(X_n) \to \mathbb{E}h(X_\infty)$, ($\mathcal{C}_h :=$ cont. points)

$\Leftrightarrow$ $\forall$ such h $\quad h(X_n) \xrightarrow{d} h(X_\infty)$.

<u>Thm</u> (Donsker's invariance principle) $(\xi_k)_{k\geq 1}$ iid

$\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 = 1$ $\Rightarrow$ $\hat{S}_n \xrightarrow{d} W$ (std Wiener). ∎

The fdd's of X_n converge weakly to those of X_∞ if $\forall t_1 < t_2 < \dots < t_k$

$$\mu^{(n)}_{t_1 \dots t_k} \overset{w}{\Longrightarrow} \mu^{(\infty)}_{t_1 \dots t_k}$$

Law$(X_n(t_1) .. X_n(t_k))$ $\quad$ Law of $(X_\infty(t_1) .. X_\infty(t_k))$

$$(X_n(t_1) .. X_n(t_k)) \xrightarrow{d} (X_\infty(t_1) \dots X_\infty(t_k))$$

By multivariate CLT

$\hat{S}_n$ converges to W in the sense of fdd's

$$\hat{S}_n(t) = \frac{1}{\sqrt{n}} S(tn) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k + \overbrace{\frac{1}{\sqrt{n}} (nt - [nt]) \xi_k}^{\to 0} \xrightarrow{d} N(0,t)$$

$\text{Law}\,(\hat{S}_n(t_1), \dots \hat{S}_n(t_k)) \xRightarrow{w} N(0, Q)$

$Q_{ij} = t_i \wedge t_k$. $\qquad Q \in \mathbb{R}^{k \times k}$

Convergence of fdd's does not imply conv in distr of process

<u>Proposition</u> Assume $(X_n)_{n \geq 1}$ unif tight in $C([0,\infty))$

If fdd's of X_n converge to those of X_∞ then $X_n \xrightarrow{d} X_\infty$

<u>Proof</u> $C([0,\infty))$ complete metric separable

Hence $\forall$ seq (n_k) $\exists$ $(\bar{n}_k) \subseteq (n_k)$ such that

$X_{\bar{n}_k} \xrightarrow{d} X_S$. $\qquad (\bar{n}_k) = S$

X_S a r.v. in $C([0,\infty))$

in part $\mathbb{E} h(X_{\bar{n}_k}) \to \mathbb{E} h(X_S)$ $\forall h$ bdd cont.

$$h(x) = \prod_{i=1}^{m} h_i(x(t_i)) \qquad h_i \in C_b(\mathbb{R}).$$

$$\mathbb{E} h(X_{\bar{n}_k}) \to \mathbb{E} h(X_\infty)$$

so $\mathbb{E} h(X_S) = \mathbb{E} h(X_\infty)$ for any such h $\Rightarrow$ hence $\forall h$ bdd cont.

$\forall (n_k)$ $\exists$ $(\bar{n}_k) = S \subseteq (n_k)$ st $\mathbb{E} h(X_{\bar{n}_k}) \to \mathbb{E} h(X_\infty)$ $\forall h$ bdd cont.

$\Rightarrow$ $\mathbb{E} h(X_n) \to \mathbb{E} h(X_\infty)$ $\qquad \square$

fdds $\hat{S}_n$ converge to fdd's of W.

Need only to prove $(\hat{S}_n)_{n\geq 1}$ unif tight

Will do it for $t\in[0,1]$ $(C([0,1]))$

Need to construct $K_\ell \subseteq C([0,1])$ with compact closure

st $\sup_n \mathbb{P}(\hat{S}_n \notin K_\ell) \to 0$ as $\ell \to \infty$.

$x : [0,1] \to \mathbb{R}$

$$Q_\delta(x) := \sup_{\substack{t,s\in[0,1] \\ |t-s|\leq\delta}} |x(t)-x(s)|$$

<u>Thm</u> (Ascoli Arzelá) $K \subseteq C([0,1])$ has compact closure iff

$$(1) \quad \sup_{x\in K} |x(0)| < \infty$$

$$(2) \quad \lim_{\delta\to 0} \underbrace{\sup_{x\in K} Q_\delta(x)}_{\overline{Q}_\delta(K)} = 0$$

<u>Thm</u> $\{\xi_k\}$ iid $\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 = 1$

$$\lim_{\delta\to 0} \sup_n \mathbb{P}\left(Q_\delta(\hat{S}_n) \geq \frac{1}{M}\right) = 0 \quad \forall M \in \mathbb{N}$$

<u>Claim</u> This implies tightness of $(\hat{S}_n)_{n\in\mathbb{N}}$ $\square$

(*) $\lim_{\delta\to 0} \sup_n \mathbb{P}\left(Q_\delta(S_n) > \frac{1}{M}\right) = 0 \quad \forall M.$

$\Rightarrow$ tightness.

Define $F_{M,\delta} := \left\{ x : x(0)=0,\ Q_\delta(x) \le \frac{1}{M} \right\}.$

- $F_{M,\delta}$ is closed.
- Choose $\delta_M \downarrow 0$ as $M\to\infty$

$$\tilde{F} = \bigcap_{M=1}^{\infty} F_{M,\delta_M}.\qquad \text{is compact.}$$

indeed
- closed (intersection of closed)
- $x\in\tilde{F}$ $\quad x(0)=0$

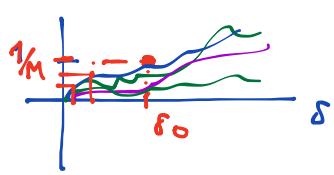
$$Q_\epsilon(x) \le Q_{\delta_{M(\epsilon)}}(x) \le \frac{1}{M(\epsilon)} \downarrow 0$$

$$M(\epsilon) = \inf\{ M \ge N : \delta_M > \epsilon \}$$

$M(\epsilon) \uparrow \infty$ as $\epsilon\to 0$

$$\sup_{x\in\tilde{F}} Q_\epsilon(x) \le \frac{1}{M(\epsilon)} \downarrow 0 \quad \text{as } \epsilon\to 0$$

F_{M,δ_0}



will define

Define $\forall \ell\in\mathbb{N}$ $\qquad \delta_M(\ell) \downarrow 0$ as $M\to\infty$

$$\tilde{F}_\ell = \bigcap_{M=1}^{\infty} F_{M,\delta_M(\ell)}$$

$$\lim_{\delta\to 0} \sup_n \mathbb{P}\left(Q_\delta(\hat{S}_n) > \frac{1}{M}\right) = 0$$

$$\sup_n \mathbb{P}(\hat{S}_n \notin F_{M,\delta}) \leq \epsilon_M(\delta) \quad \downarrow 0 \text{ as } \delta \downarrow 0$$

$$\sup_n \mathbb{P}(\hat{S}_n \notin \tilde{F}_\ell) \leq \sup_n \sum_{M=1}^{\infty} \mathbb{P}(\hat{S}_n \notin F_{M,\delta_M(\ell)}) = (*)$$

Choose $\delta_M(\ell)$ such that

$$\sup_n \mathbb{P}(\hat{S}_n \notin F_{M,\delta_M(\ell)}) \leq 2^{-\ell-M}.$$

$$(*) \leq \sum_{M=1}^{\infty} 2^{-\ell-M} \leq 2^{-\ell} \qquad \downarrow 0 \text{ as } \ell\to\infty.$$

$$\lim_{\ell\to\infty} \sup_n \mathbb{P}(\hat{S}_n \notin \tilde{F}_\ell) = 0$$

$$\hat{S}_n(t) = \frac{1}{\sqrt{n}} S(nt) \qquad\qquad \tilde{S}_n(t) = \frac{1}{n^{0.51}} S(nt)$$

$$\mathbb{E}\left[\frac{S(nt)^2}{n}\right] \sim 1 \qquad\qquad \mathbb{E}\,\tilde{S}(n,t) \approx \frac{nt}{n^{1.02}} \downarrow 0$$

$$\tilde{S}_n(t) \to \tilde{S}_\infty(t) = 0$$

$\forall h$ ~~bdd~~ cont $\forall (n_k)\ \exists (\bar{n}_k) \subseteq (n_k)$ st $\mathbb{E}h(X_{\bar{n}_k}) \to \mathbb{E}h(X_\infty)$ (*)

Want to show $\underbrace{\mathbb{E}h(X_n)}_{z_n} \to \mathbb{E}h(X_\infty)$

assume false

eg $\limsup \mathbb{E}h(X_n) \to \bar{x} \neq \mathbb{E}h(X_\infty)$, $\bar{x} \in \overline{\mathbb{R}}$

$\exists (n_k)$ st $\mathbb{E}h(X_{n_k}) \to \bar{x} \neq \mathbb{E}h(X_\infty)$

By(*) $\exists (\bar{n}_k) \subseteq (n_k)$ st $\mathbb{E}h(X_{\bar{n}_k}) \to \mathbb{E}h(X_\infty)$

X, Y two sp

Assume $\forall t_1 \dots t_k\ \forall\ \bar{h}(x) = \prod_{i=1}^{k} h(x(t_i))$

$\mathbb{E}\bar{h}(X) = \mathbb{E}\bar{h}(Y)$ $\qquad \mathbb{P}(X \in C) = \mathbb{P}(Y \in C)$

then the law of X = law of Y

(as prob measures on $\mathcal{B}^{[0,\infty)}$)

$\Rightarrow$ $\mathbb{E}h(X) = \mathbb{E}h(Y)$ $\forall h$ bdd $h \in m\mathcal{B}^{[0,\infty)}$

In this proof wanted to say

$\forall h$ cont on $C([0,\infty))$ (is in $m\mathcal{B}^{[0,\infty)}$)

✓ $\mathcal{B}^{[0,\infty)}$ restricted to $C([0,\infty))$

is $\mathcal{B}_{C([0,\infty))}$ ✓ $\qquad h(x) = \max_{t \in [0,1]} x(t)$

not in $m\mathcal{B}^{[0,1]}$

Cont on $C([0,\infty])$

Take $h(x) = \sup_{t \in [0,1]} x(t)$ cont on $C([0,1])$

We proved

$$\mathbb{E} \prod_{i=1}^{k} h_i(X(t_i)) = \mathbb{E} \prod_{i=1}^{k} h_i(Y(t_i))$$

we cannot conclude that

$$\mathbb{E}\, h(X) = \mathbb{E}\, h(Y)$$

$X = 0$ ident $\qquad Y(t) = \begin{cases} 1 & \text{if } t = U \\ 0 & \text{ow} \end{cases}$

$U \sim \mathrm{Unif}([0,1])$

$X(t) =$

Fix $\Pi \subseteq [0,1]$ dense $\quad \forall t$ seq $t_k(t) \to t$
$x \in C([0,1])$: $\qquad t_k$

$$\mathscr{C} = \{ \pi_{t_1 \ldots t_k}(x) \in (A_1 \times \ldots \times A_k) : t_1 \ldots t_k \in [0,1], \; A_1 \ldots A_k \in \mathcal{B}_{\mathbb{R}}, \; k \in \mathbb{N} \}$$

$$\mathscr{C} \subseteq 2^{C([0,1])}$$

$$\sigma(\mathscr{C}) = \mathcal{B}_{C([0,1])} \longleftarrow (*)$$

$$\bar{\mathscr{C}} = \{ x \in 2^{[0,1]} \; \pi_{t_1 \ldots t_k}(x) \in A_1 \times \ldots \times A_k \ldots \}$$

$$\sigma(\bar{\mathscr{C}}) = \mathcal{B}^{[0,1]}$$

$\mathcal{B}^{[0,1]}$ restricted to $C([0,1])$ is $\mathcal{B}_{C([0,1])}$