<u>Lecture 13</u> We still need to prove tightness

$$Q_{\delta,T}(x) = \sup_{\substack{0\le t,s\le T \\ |t-s|\le\delta}} |x(t)-x(s)|$$

<u>Thm</u> $\hat{S}_n(t) = n^{-1/2} S(nt)$ $\qquad S(t) :=$ random walk

$$\lim_{\delta\to 0} \sup_n \underbrace{\mathbb{P}\left(Q_{\delta,1}(\hat{S}_n) \ge \frac{1}{M}\right)}_{p_{n,\delta}} \; 0 \qquad \forall\, M<\infty \qquad \square$$

<u>Proof</u>

I know: $\boxed{\lim_{\delta\to 0} p_{n,\delta} = 0}$

decreasing

$$Q_{\delta,1}(\hat{S}_n) \le \delta\cdot n \max_{0\le \ell\le \lfloor n\rfloor} |S(\ell+1)-S(\ell)|$$

$$\le \delta n \max_{\ell\le\lfloor n\rfloor} |\xi_\ell|$$

hence $\lim_{\delta\to 0} \sup_{n\le n_0} p_{n,\delta} = 0 \quad (*) \quad \forall\, n_0<\infty$

I want to show $\lim_{\delta\to 0} \overline{\limsup_n p_{n,\delta}} = 0$

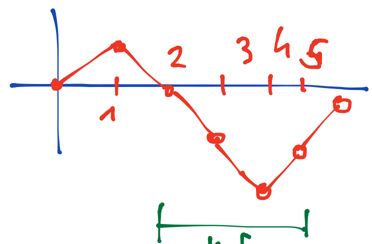
Sufficient to show $\lim_{\delta\to 0} \limsup_{n\to\infty} p_{n,\delta} = 0$

[Why? $\quad b(n_0) = \inf_{\delta>0} \sup_{n\ge n_0} p_{n,\delta}$

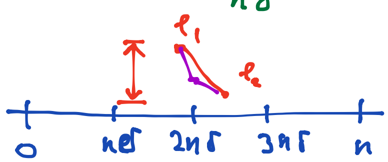
Wts $b(1) = 0$ but I know $b(n_0)$ indep of n_0, hence I need $\lim_{n_0\to\infty} \left(= \inf_{n_0}\right) \inf_\delta \sup_{n\ge n_0} p_{n,\delta} = 0$

$$p_{n,\delta} = \mathbb{P}\left(Q_{\delta,1}(\hat{S}_n) \geq \frac{1}{M}\right)$$

$$= \mathbb{P}\left(Q_{n\delta,n}(S) \geq \frac{\sqrt{n}}{M}\right)$$



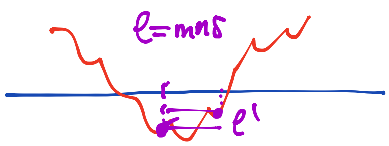
$$(*) \quad = \mathbb{P}\left(\max_{\substack{0 \leq \ell_1, \ell_2 \leq n \\ |\ell_1 - \ell_2| \leq n\delta}} |S_{\ell_1} - S_{\ell_2}| \geq \frac{\sqrt{n}}{M}\right)$$



$$\leq \mathbb{P}\left(\max_{\substack{\ell \in N \\ \ell' \in [\ell, \ell + n\delta]}} |\underbrace{S_\ell - S_{\ell'}}_{\overset{d}{=}\, S_{\ell' - \ell}}| \geq \frac{\sqrt{n}}{3M}\right)$$

$N = \{0, n\delta, 2n\delta, \dots n\}$

$$\leq |N| \cdot \mathbb{P}\left(\max_{0 \leq k \leq n\delta} |S_k| \leq \frac{\sqrt{n}}{3M}\right) \quad \checkmark$$



$$\left(\sum_{\ell \in N} \mathbb{P}\left(\max_{\ell' \in [\ell, \ell + n\delta]} |S_{\ell'} - S_\ell| \geq \frac{\sqrt{n}}{3M}\right)\right)$$

$$\leq 100\left(1 + \frac{1}{\delta}\right) 2 \cdot \mathbb{P}\left(|S_{n\delta}| \geq \frac{\sqrt{n}}{3M}\right) \qquad \delta < 1$$

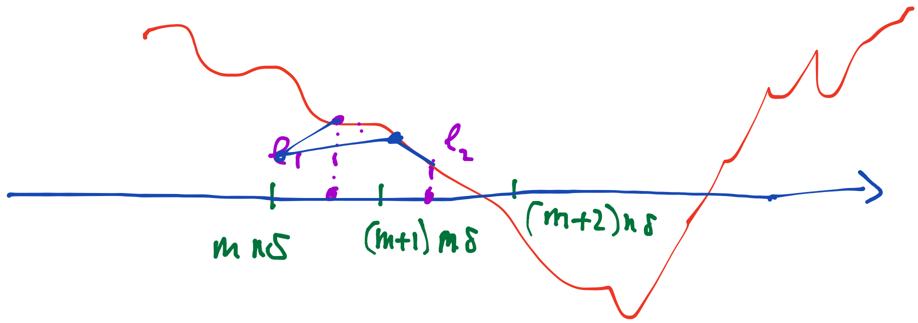
$$p_{n,\delta} \leq \frac{C}{\delta} \mathbb{P}\left(|S_{n\delta}| > \frac{\sqrt{n}}{3M}\right)$$

$$\limsup_{n \to \infty} p_{n,\delta} \leq \frac{C}{\delta} \mathbb{P}\left(|G| \geq \frac{1}{\sqrt{\cancel{n}\delta}} \frac{\sqrt{\cancel{n}}}{3M}\right)$$

$$\leq \frac{C'}{\delta} \exp\left(-\frac{1}{18M^2\delta}\right)$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} p_{n,\delta} = 0.$$

$$(\text{UB on } (*)) \quad n^2\delta \max_{|\ell_1 - \ell_2| \leq n\delta} \overbrace{\mathbb{P}\left(|S_{\ell_1} - S_{\ell_2}| \geq c\sqrt{n}\right)}^{n \mapsto a} \to C(\delta)$$



$$\max_{\ell\in N}\ \max_{\ell'\in[\ell,\ell+n\delta]} |S_\ell - S_{\ell'}| = Z$$

Case 1

$$|S_{\ell_1}-S_{\ell_2}| \le |S_{\ell_1}-S_{mn\delta}| + |S_{\ell_2}-S_{mn\delta}| \le 2Z$$

Case 2

$$\le |S_{\ell_1}-S_{m\delta}| + |S_{\ell_2}-S_{(m+1)\delta n}| + |S_{(m+1)\delta n}-S_{m\delta n}| \le 3Z$$

$$\mathbb{P}\Big(\max_{|\ell_1-\ell_2|\le n\delta} |S_{\ell_1}-S_{\ell_2}| \ge \frac{\sqrt{n}}{M}\Big) \le \mathbb{P}\Big(3Z \ge \frac{\sqrt{n}}{M}\Big).$$

Donsker : Translate BM results $\Rightarrow$ RW results.

Example (1) $$M_{n,t} := \frac{1}{\sqrt{n}} \max_{0\le k\le nt} \sum_{\ell=1}^{k} \xi_\ell$$

$$M_t := \max_{0\le s\le t} W_s = g(W_\cdot)$$

$g(x) = \max_{s\in[0,t]} x(s)$ $\qquad$ g cont on $C([0,\infty))$

$$M_{n,t} = g(\hat{S}_n)$$

$$M_{n,t} \xrightarrow{d} M_t .$$

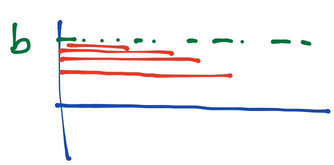
(2) $$\hat{\tau}_{b,n} = \frac{1}{n} \inf\Big\{ k\ge 1 : \sum_{\ell=1}^{k} \xi_\ell \ge b\sqrt{n} \Big\}$$

$$\tau_b := \inf\{ t : W_t \ge b \}$$

$\tau_b = g(W)$; $\quad g(x) = \inf\{t : x(t)\ge b\}$

$$\hat{\tau}_{b,n} = g(\hat{S}_n)$$

g not continuous on $C([0,\infty))$

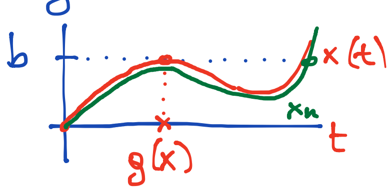


$x_n(t) = b - \frac{1}{n}$ $\qquad$ $x_\infty(t) = b$

$x_n \to x_\infty$

D_g

$g(x_n) = \infty$ $\qquad$ $g(x_\infty) = 0$



$x_n(t) = x(t) - \frac{1}{n}$

$\tau_{(b,\infty)} = \tau_{[b,\infty)}$ a.s for W.

$$X_n \xrightarrow{d} X_\infty \qquad \mathbb{P}(X_\infty \in D_g) = 0$$

$$\Rightarrow \quad g(X_n) \xrightarrow{d} g(X_\infty).$$

<u>Theorem</u> (Kolmogorov - Smirnov) $(X_i)_{i \geq 1}$ iid $\stackrel{d}{=} X$

F_X distr function continuous. $\quad F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{x \geq X_i}$.

$$\sqrt{n} \sup_{x \in \mathbb{R}} |F_n(x) - F_X(x)| \xrightarrow{d} \sup_{t \in [0,1]} \hat{W}_t$$

$$\hat{W}_t := W_t - t W_1 \qquad \text{std Brownian bridge.}$$

$$\left[Z_{n,t} = \sqrt{n} \left[F_X^{-1}(F_n(x)) - x \right] \right]_{0 \leq t \leq 1} \xrightarrow{d} \left(\hat{W}_t \right)_{t \in [0,1]}.$$

□

<u>Alternative approach</u> : Skorokhod representation

<u>Thm</u> $(W_t, \mathcal{F}_t)$ std Wiener.

$U_1, U_2 \underset{iid}{\sim} \mathrm{Unif}([0,1])$ indep. of $\mathcal{F}_\infty$.

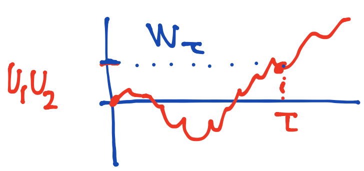
$\mathcal{G}_t = \sigma(\mathcal{F}_t \cup \sigma(U_1, U_2))$

P_X law of random var X $\quad \mathbb{E}X = 0, \ \mathbb{E}|X| < \infty.$

Then $\exists$ $\mathcal{G}_t$- stopping time τ, a.s. finite.

st
$$W_\tau \overset{d}{=} X$$

Further $\mathbb{E}\tau = \mathbb{E}X^2 \qquad \mathbb{E}\tau^2 \le 2\mathbb{E}(X^4)$ $\square$



<u>Basic example</u> $X \in \{-a, b\}$

$$\mathbb{E}(X) = 0 \Rightarrow \mathbb{P}(X = b) = \frac{a}{a+b}, \quad \mathbb{P}(X = -a) = \frac{b}{a+b}$$

$\tau := \inf\{ t \geq 0 : t \notin (a, b)\}$

$\Rightarrow \quad W_\tau \overset{d}{=} X \quad$ by OST.

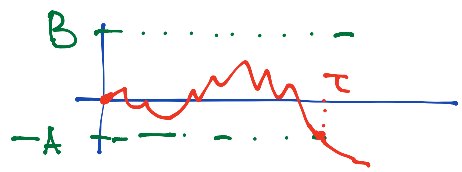
General case :

"decompose X into 2-points rv's"

Given P_X, construct random vector $(A,B) \in \mathbb{R}^2_{\geq 0}$

st if

$$\tilde{X} = \begin{cases} -A & \text{w prob } \frac{B}{A+B} \\ B & \text{w prob } \frac{A}{A+B} \\ 0 & \text{if } A+B=0 \end{cases}$$



then $\tilde{X} \sim P_X$

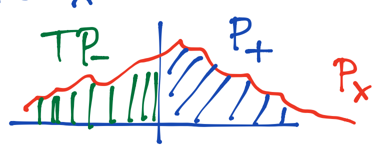
Can construct $(A,B) \in m\sigma(U_1, U_2)$ and set

$$\tau := \inf\{t \geq 0 : W_t \notin (-A, B)\}.$$

By OST $\quad W_\tau \overset{d}{=} X$

Need to construct $P_{A,B}$. $\quad T\mu((x_1, x_2]) = \mu([-x_2, -x_1))$

$$P_+ = I_{x>0} P_X \qquad P_- = T(I_{x<0} P_X)$$



Naive idea $\quad P_{A,B} = P_- \otimes P_+ + \delta_{0,0} P_X(\{0\})$

$$\Rightarrow P_{A,B} = \delta_{00} P_X(\{0\}) + \frac{a+b}{m_X} P_- \otimes P_+ \qquad \checkmark$$

$$m_X = \mathbb{E}|X|/_2 = \left[\int x P_+(dx) + \int x P_-(dx)\right]/_2$$

$$P_{A,B}(da,db) = \delta_{00}(da,db)\,P_A(\{0\}) + \underbrace{\frac{a+b}{m_X}\; P_-\otimes P_+(da,db)}$$

Need to check $\tilde{X} \overset{d}{=} X$

$$\mathbb{E}[f(\tilde{X})] = \mathbb{E}\left\{ \frac{B}{A+B} f(-A) + \frac{A}{A+B} f(B) \right\}$$

$$\overset{\text{using } \checkmark}{=} \mathbb{E} f(X).$$

Need to prove bds on $\mathbb{E}\tau$, $\mathbb{E}\tau^2$.

$$\tau_{ab} = \inf\{ t \geq 0 : W_t \notin (a,b) \}$$

$$\mathbb{E}\,\tau_{ab} = ab \Rightarrow \mathbb{E}\,\tau = \mathbb{E}[AB] = \mathbb{E}(X^2)$$

$$\mathbb{E}(\tau_{ab}^2) = (ab)^2 + ab\,(a^2+b^2)/3 \qquad a^2b^2 \leq ab(a^2+b^2)$$

$$\leq C'ab\,(a^2+b^2) \leq C\,ab\,(a^2+b^2-ab)$$

$$\mathbb{E}\tau \leq C\int_{\mathbb{R}^2_{\geq 0}} ab\,(a^2+b^2-ab)\;P_{AB}(da,db)$$

$$\leq \frac{C}{m_X}\int_{\mathbb{R}^2_{\geq 0}} (ab)\,\underbrace{(a^2+b^2-ab)}\;\underbrace{(a+b)}\;P_-(da)\,P_+(db)$$

$$\leq \frac{C}{m_X}\int_{\mathbb{R}_{\geq 0}} (a^4 b + a b^4)\;P_-(da)\,P_+(db)$$

$$\leq \frac{C}{\mathbb{E}|X|}\left\{ \mathbb{E}(X_+^4)\,\mathbb{E}(X_-) + \mathbb{E}(X_-^4)\,\mathbb{E}(X_+) \right\}$$

$$\leq C\left\{ \mathbb{E}(X_+^4) + \mathbb{E}(X_-^4) \right\} \leq C\,\mathbb{E}(X^4).$$