Lecture 14

Thm $(W_t, \mathcal{F}_t)$ std Wiener process, $U_1, U_2 \underset{iid}{\sim} \mathrm{Unif}([0,1])$
$\mathcal{F}_\infty$ indep of $\sigma(U_1, U_2)$, $\mathcal{G}_t := \sigma(\mathcal{F}_t, \sigma(U_1, U_2))$.
P_X law of X, $\mathbb{E}X = 0$, $\mathbb{E}|X| < \infty$.
Then there exists τ, $\mathcal{G}_t$ stopping time $\mathbb{P}(\tau < \infty) = 1$
st

$$W_\tau \overset{d}{=} X$$

and $\mathbb{E}\tau = \mathbb{E}X^2$, $\mathbb{E}\tau^2 \leq 2\mathbb{E}X^4$.

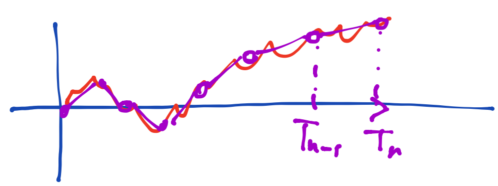
Thm $(\xi_k)_{k\geq 1}$ iid $\mathbb{E}\xi_k = 0$ $\mathbb{E}|\xi_k| < \infty$, $S_n = \sum_{k=1}^n \xi_k$

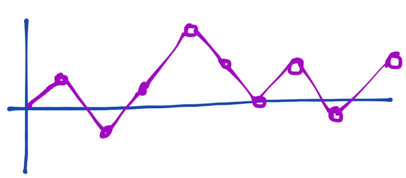
$(W_t, \mathcal{F}_t)_{t\geq 0}$ Wiener $(U_k)_{k\geq 1}$ iid $\mathrm{Unif}([0,1])$.

$\sigma((U_k)_{k\geq 0})$ indep of $\mathcal{F}_\infty$. $\mathcal{F}_{k,t} = \sigma(\mathcal{F}_t, \sigma((U_i)_{i\leq 2k}))$

Then there exist $(T_k)_{k\geq 1}$ T_k is $\mathcal{F}_{k,t}$ s.time such that defining $\tilde{S}_n := W_{T_n}$

$$(\tilde{S}_n)_{n\geq 1} \overset{d}{=} (S_n)_{n\geq 1}$$





Proof Construct $\{(A_k, B_k)\}_{k\geq 1}$
where $(A_k, B_k) = F(U_{2k-1}, U_{2k})$
such that

$$\tau_{A_k B_k} := \inf\{t : W_t \notin (-A_k, B_k)\}$$

$$W_{\tau_{A_k B_k}} \overset{d}{=} \xi_1 \qquad (\mathbb{E}\xi_1 = 0.\ \mathbb{E}|\xi_1| < \infty)$$

Construct (T_k) recursively $T_0 = 0$, $T_1 \ldots T_k$

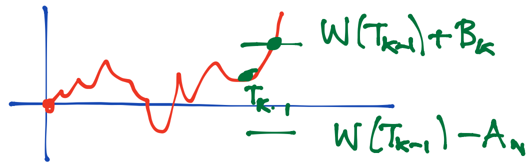
$$T_k = T_{k-1} + \tau_k$$

$$\tau_k := \inf\{t \geq 0 : W(T_{k-1}+t) - W(T_{k-1}) \notin (-A_k, B_k]\}$$

By strong Markov prop

$$W(\underbrace{T_{k-1}+\tau_k}_{T_k}) - W(T_{k-1}) \overset{d}{=} W(\tau_k) \overset{d}{=} \xi_k$$

indep of $\mathcal{F}_{k-1,t}$.



<u>Thm</u> (Strassen) Let

(1) $(M_k)_{k\geq 1}$ a MG wrt $\mathcal{G}_k = \sigma((M_i)_{i\leq k})$ $M_0 = 0$, $D_k := M_k - M_{k-1}$

(2) $(U_i)_{i\geq 1}$ iid $\sim Unif([0,1])$

(3) $(W_t, \mathcal{F}_t^W)$ std Wiener process. $\mathcal{G}_t = \sigma(\mathcal{F}_t^W, \sigma(U_i)_{i\geq 1})$

With (2) and (3) indep. $\mathcal{F}_{k,t} := \sigma(\mathcal{F}_t^W, \sigma((U_i)_{i\leq 2k}))$

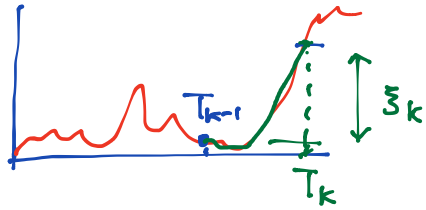
(notice that $(\mathcal{F}_{k,t}, W_t)$ strong Markov Wiener)

Then $\exists$ for each k an $\mathcal{F}_{k,t}$ stopping time T_k st letting $\mathcal{H}_k = \mathcal{F}_{k,T_k}$, $\tau_k := T_k - T_{k-1}$ we have

(1) $(W_{T_k})_{k\geq 1}$ has same distr as $(M_k)_{k\geq 1}$.

(2) $$\mathbb{E}(\tau_k \mid \mathcal{H}_{k-1}) = \overbrace{\mathbb{E}(D_k^2 \mid M_0, \dots M_{k-1})}^{F_k(M_0,\dots M_{k-1})}\Big|_{\substack{M_\ell = W(T_\ell)\\ \ell \leq k-1}}$$

$$\mathbb{E}(\tau_k^2 \mid \mathcal{H}_{k-1}) \leq 2\,\mathbb{E}(D_k^4 \mid M_0, .. M_{k-1})\Big|_{\substack{M_\ell = W(T_\ell)\\ \ell \leq k-1}}$$

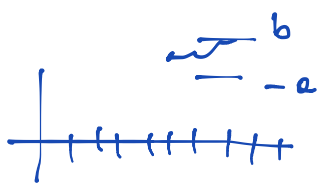


$$\mathcal{G}_t := \sigma(\mathcal{F}^W_t, \sigma((\xi_\ell)_{\ell \ge 1}))$$

$$\tilde{\tau}_k = \inf\{ t : W_{T_{k-1}+t} - W_{T_{k-1}} = \xi_k \}$$

$\tilde{T}_k = \tilde{T}_{k-1} + \tilde{\tau}_k$. $\tilde{T}_k$ is a $\mathcal{G}_t$ st. $\mathbb{E}\tau_b = \lim_{a\to\infty} \mathbb{E}\tau_{ab}$ ✓

$\mathbb{E}\,\tilde{\tau}_k = \infty$.



$\boxed{\mathbb{E}\tau_{ab} = ab}$

$W_t^2 - t$

$\mathbb{E} W_\tau^2 = \mathbb{E}\tau$

$T_k \approx ck$

Proof. Construct (T_k) recursively. Assume $T_1, \dots T_{k-1}$ given. $((W_t, \mathcal{F}_{k-1,t})$ strong Markov$)$

$W_{T_{k-1}+t} - W_{T_{k-1}}$ std Wiener

independent of $\mathcal{F}_{k-1,T_{k-1}} = \mathcal{H}_{k-1}$

$$\mathbb{P}(D_k \le x \mid M_0, \dots M_{k-1}) = F_k(x; M_0, \dots M_{k-1})$$

$$F_k^*(x) = F_k(x; W_{T_0}, \dots W_{T_{k-1}})$$

[Skhorokhod constr: given distr F_X, $U_1, U_2 \overset{iid}{\sim}$ Unif

$$(A, B) = f(U_1, U_2; F_X)$$

st $\mathbb{P}(W_{\tau_{AB}} \le y) = F_X(y)$ $\forall y$]

$$(A_k, B_k) = f(U_{2k-1}, U_{2k}; \underset{W_{T_0}, \dots W_{T_{k-1}}}{F_k^*})$$ (same f)

F_k^* is a random function uncountably many values $(F_k^*(x))_{x\in\mathbb{R}}$

Q: Is $f(U_{2k-1}, U_{2k}; F_k^*) \in m\mathcal{F}_{k,T_{k-1}}$?

A: Yes. (see notes).

$\tau_k := \inf\{t : W(T_{k-1}+t) - W(T_{k-1}) \notin (-A_k, B_k)\}$

$T_k := T_{k-1} + \tau_k$ $\quad [\mathbb{E}[h_k(W_{T_k}-W_{T_{k-1}})\, h_{k-1}(W_{T_{k-1}}-W_{T_{k-2}})\dots]$

$\mathbb{E}[\mathbb{E}(h_k(\,)|\mathcal{H}_{k-1})\, h_{k-1}\dots]$

$\mathbb{P}(W_{T_k}-W_{T_{k-1}}\le x\,|\,\mathcal{H}_{k-1}) = \mathbb{P}(D_k\le x\,|\,M_0\dots M_{k-1})$ $\quad$ (*)

$M_\ell = W_{T_\ell}$, $\ell\le k-1$.

$\mathbb{E}(\tau_k\,|\,\mathcal{H}_{k-1}) = \mathbb{E}(D_k^2\,|\,M_0\dots M_{k-1})\big|\dots$

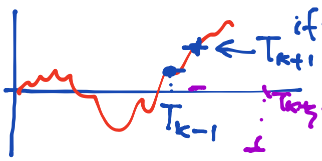
$\mathbb{E}(\tau_k^2\,|\,\mathcal{H}_{k-1}) \le 2\,\mathbb{E}(D_k^4\,|\,M_0\dots M_{k-1})\big|\dots$

Need to check $\forall k$

$(D_1,\dots D_k) \overset{d}{=} (W_{T_1}-W_0,\dots, W_{T_k}-W_{T_{k-1}})$

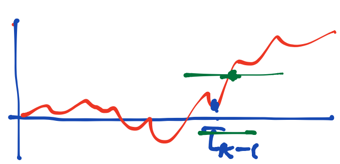
by induction using (*) $\square$

Example RW $(\xi_k)_{k\ge1}$ iid $\xi_k\sim \mathrm{Unif}(\{\pm1\})$

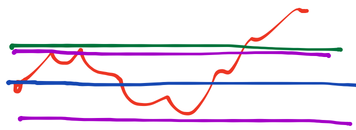


if $\xi=+1$ generate ξ_k's $\in\{\pm1\}$

T_{k+1} $\xi=-1$ $\quad \tau_k = \inf\{t: W_{T_{k-1}+t} - W_{T_k} = \xi_k\}$



$(A_k,B_k)=(+1,+1)$



ξ

$\tau_B = \tau_\xi = \inf\{t : W_t = \xi\}$

$\xi \sim \mathrm{Unif}(\pm 1)$ $\qquad \tau_S = \inf\{t \,;\, W_t \notin (-1,+1)\}$

$$\mathbb{P}(\tau_B < \infty) = \mathbb{P}(\tau_S < \infty) = 1$$

$$W_{\tau_B} \overset{d}{=} W_{\tau_S} \sim \mathrm{Unif}(\{\pm 1\})$$

$$\mathbb{E}\tau_B = \infty \qquad \mathbb{E}\tau_S = 1.$$

<u>Thm</u> (MG CLT) $\forall n \geq 1$ $(M_k^{(n)}, \mathcal{F}_k^{(n)})$ a L^2 MG (Canonical filtration)

$M_0^{(n)} = 0$. $D_k^{(n)} := M_k^{(n)} - M_{k-1}^{(n)}$.

$$\langle M^{(n)} \rangle_k := \sum_{\ell=1}^{k} \mathbb{E}\left[D_\ell^{(n)\,2} \mid \mathcal{F}_{k-1}^{(n)} \right].$$

$\leftarrow \mathbb{E}[\tau_k^{(n)} | \dots]$

<u>Assume</u>

(1) $\forall t \in [0,1]$ $\qquad \langle M^{(n)} \rangle_{\lfloor nt \rfloor} \xrightarrow{p} t$.

(2) $\forall \epsilon > 0$

$$g_n(\epsilon) := \sum_{k=1}^{n} \mathbb{E}\left[D_k^{(n)\,2} \mathbb{1}_{|D_k^{(n)}| \geq \epsilon} \mid \mathcal{F}_{k-1}^{(n)} \right] \xrightarrow{P} 0$$

<u>Then</u> $\qquad \hat{S}_n(t) := M_{\lfloor nt \rfloor}^{(n)} + (nt - \lfloor nt \rfloor) D_{\lfloor nt \rfloor + 1}^{(n)}$

Converges in distr (in $C([0,1])$) to std Wiener $(W_t)_{t \in [0,1]}$

$\mathbb{E}$

Donsker is special case $\quad M_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{\ell=0}^{k} \xi_\ell \qquad \xi_\ell$ iid

$D_k^{(n)} = \xi_k / \sqrt{n}$

$$\langle M^{(n)}\rangle_k = \sum_{\ell=1}^{k} \mathbb{E}(D_k^{(n)2} \mid \mathcal{F}_{k-1}^{(n)}) = \sum_{\ell=1}^{k} \mathbb{E}\left(\frac{\xi_\ell^2}{n}\right) = \frac{k}{n}.$$

$$\langle M^{(n)}\rangle_{\lfloor nt\rfloor} = \frac{\lfloor nt\rfloor}{n} \xrightarrow{P} t.$$

$$g_n(\epsilon) = \sum_{\ell=1}^{n} \mathbb{E}[D_\ell^{(n)2} 1_{|D_\ell^{(n)}|\geq\epsilon} \mid \mathcal{F}_{\ell-1}^{(n)}]$$

$$= \frac{1}{n}\sum_{\ell=1}^{n} \mathbb{E}(\xi_\ell^2 1_{|\xi_\ell|\geq\epsilon\sqrt{n}}) = \mathbb{E}(\xi_1^2 1_{|\xi_1|\geq\epsilon n})$$

$\to 0$ by DOM.

$X_1 .. X_\ell \in m\mathcal{H}_\ell \quad \mathcal{H}_\ell \subseteq \mathcal{H}_{\ell+1} ..$

$(X_1, \dots X_k) \quad (Y_1, \dots Y_k) \quad \mathbb{R}^k$ r. vectors.

$$\mathbb{P}(X_\ell \leq t \mid \mathcal{H}_\ell) = F_{X_\ell|X_1..X_{\ell-1}}(t; X_1 .. X_{\ell-1}) \quad (\#)$$

$$\mathbb{P}(Y_\ell \leq t \mid Y_1 .. Y_{\ell-1}) = F_{Y_\ell|Y_1..Y_\ell}(t; Y_1 .. Y_{\ell-1})$$

$$F_{X_\ell|X_1..Y_{\ell-1}} = F_{Y_\ell|Y_1..Y_{\ell-1}}$$

$$(\#) \Rightarrow \mathbb{P}(X_\ell \leq t \mid X_1 .. X_{\ell-1}) = F_{X_\ell|X_1..X_{\ell-1}}(t; X_1 .. X_{\ell-1})$$

By induction over ℓ $P_{X_1..X_\ell} = P_{Y_1..Y_\ell}$. Suff to show

$$\mathbb{E}[h_1(X_1) .. h_\ell(X_\ell)] = \mathbb{E}[h_1(Y_1) .. h_\ell(Y_\ell)] \quad (\#\#)$$

$$LHS = \mathbb{E}[h_1(X_1) .. h_{\ell-1}(X_{\ell-1}) \underbrace{\mathbb{E}[h_\ell(X_\ell) \mid X_1 .. X_{\ell-1}]}_{g_\ell^k(X_1..X_{\ell-1})}]$$

$$= \mathbb{E}(f(X_1 .. X_{\ell-1})) = \mathbb{E}(f(Y_1 .. Y_{\ell-1}))$$