Lecture 15

Theorem (MG CLT) $\forall n \geq 1$, $(M_k^{(n)}, \mathcal{F}_k^{(n)})$ an L^2 MG
with $M_0^{(n)} = 0$, $D_k^{(n)} := M_k^{(n)} - M_{k-1}^{(n)}$

$$\langle M^{(n)} \rangle_k := \sum_{\ell=1}^{k} \mathbb{E}\{ D_\ell^{(n)2} \mid \mathcal{F}_{\ell-1}^{(n)} \}$$

Assume that (1) $\langle M^{(n)} \rangle_{\lfloor nt \rfloor} \xrightarrow{P} t \quad \forall t \in [0,1]$

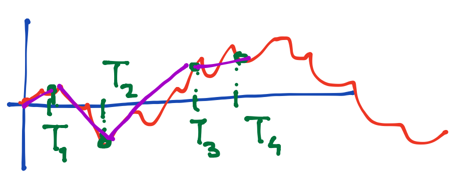
(2) $$g_n(\epsilon) = \sum_{k=1}^{n} \mathbb{E}[D_k^{(n)2} \mathbb{1}_{|D_k^{(n)}| \geq \epsilon} \mid \mathcal{F}_{k-1}^{(n)}] \xrightarrow{P} 0$$

then $\hat{S}_n(t) := M_{\lfloor nt \rfloor}^{(n)} + (nt - \lfloor nt \rfloor) D_{\lfloor nt \rfloor + 1}^{(n)}$

converges in distr to $(W_t)_{t \in [0,1]}$.

Lemma $(W_t)_{t \geq 0}$ Wiener $(T_{n,k})_{k \geq 1}$ non decreasing st $T_{n,\lfloor nt \rfloor} \xrightarrow{P} t \ \forall t \in [0,e]$. Define

$$\hat{S}_n(t) = W(T_{n,\lfloor nt \rfloor}) + (nt - \lfloor nt \rfloor)\left(W(T_{n,\lfloor nt \rfloor + 1}) - W(T_{n,\lfloor nt \rfloor}) \right).$$



$$\hat{S}_n\left(\tfrac{k}{n}\right) = W(T_{n,k})$$

Then we have $\|\hat{S}_n - W\|_{C[0,e]} \xrightarrow{P} 0$.

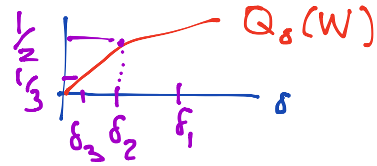
Rmk If $d(X_n, X_\infty) \xrightarrow{P} 0$ then $X_n \xRightarrow{d} X_\infty$
(ie $\mathbb{E}h(X_n) \to \mathbb{E}h(X_\infty) \quad \forall h \in C_b$). $\square$

Proof of Lemma $\ell = 1$

$$Q_\delta(x) = \sup_{\substack{0 \le t,s \le 1 \\ |t-s| \le \delta}} |x(t) - x(s)|$$

Fix ϵ small. $\forall\, k \in \mathbb{N}$, choose δ_k so that

$$\mathbb{P}\left(Q_{\delta_k}(W) > 1/k\right) \le \epsilon .$$



$$\lim_{\delta \to 0} Q_\delta(W) = 0$$

$$\lim_{\delta \to 0} \mathbb{P}\left(Q_\delta(W) > 1/k\right) = 0$$

$\exists\, n_0(k,\epsilon)$ st $\forall\, n > n_0$

$$\mathbb{P}\left(\sup_{\substack{t \in [0,1] \\ b \in \{0,1\}}} |T_{n,\lfloor nt \rfloor + b} - t| > \delta_k\right) < \epsilon \qquad (*)$$

(for t fixed $\lim_{n \to \infty} \mathbb{P}\left(|T_{n,\lfloor nt \rfloor} - t| > \delta_k\right) = 0$

naively $\mathbb{P}\left(\sup_{t \in [0,1]} |T_{n,\underbrace{\lfloor nt \rfloor}_{k}} - t| > \delta_k\right) \lesssim$

$$\lesssim n \sup_{t \in [0,1]} \underbrace{\mathbb{P}\left(|T_{n,\lfloor nt \rfloor} - t| > \delta_k\right)}_{\to 0} \not\to 0$$

In reality $s_0\, s_1, \dots s_M$ $\quad s_i = i\delta_k/100 \quad M \le \frac{100}{\delta_k}$

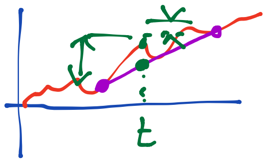
$$\mathbb{P}\left(\sup_{t \in [0,1]} |T_{n,\lfloor nt \rfloor} - t| \ge \delta_k\right) \le \mathbb{P}\left(\sup_{i \le M} |T_{n,\lfloor ns_i \rfloor} - s_i| \ge \frac{\delta_k}{3}\right)$$

$\to 0$)

With prob $\geq 1-2\epsilon$

$$Q_{\delta_k}(W) \leq 1/k \; ; \quad \sup_{\substack{t\in[0,1] \\ b\in\{0,1\}}} |T_{n,\lfloor nt\rfloor+b} - t| \leq \delta_k .$$

$$|\hat{S}_n(t) - W(t)| \leq \max_{b\in\{0,1\}} |W(T_{n,\lfloor nt\rfloor+b}) - W(t)|$$



$$\leq Q_{\delta_k}(W) \leq 1/k$$

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |\hat{S}_n(t) - W(t)| > \frac{1}{k} \Big) \leq 2\epsilon . \qquad \forall n > n_0(\epsilon)$$

$$\lim_{n\to\infty} \underbrace{\mathbb{P}\Big(\sup_{t\in[0,1]} |\hat{S}_n(t) - W(t)| > 1/k \Big) = 0}_{\|\hat{S}_n - W\|_{C[0,1]} \xrightarrow{P} 0} . \qquad \forall k$$

$\blacksquare$

<u>Proof of MG CLT</u> $\underbrace{\sigma((M_e^{(n)})_{l\le k})}(M_k^{(n)}, \mathcal{F}_k^{(n)})$, $(W_t)_{t\in[0,1]}$

Will assume stronger conditions $\quad t\in[0,1]$

$$\langle M^{(n)}\rangle_{\lfloor nt\rfloor} \xrightarrow{p} t \quad ; \quad \langle M^{(n)}\rangle_{\lfloor nt\rfloor} \le 2 \quad \text{a.s.}$$

✓ $\max_{k\le n} |D_k^{(n)}| \le \epsilon_n$ for some $\epsilon_n \downarrow 0$.

$(\Rightarrow g_n(\epsilon) = 0 \ \forall\ \epsilon_n < \epsilon)$

Can construct $\forall\, n$ $\mathcal{G}_{k,t} = \sigma(\sigma(\{U_i\}_{i\le 2k}), \mathcal{F}_t^W)$

and T_k $\mathcal{G}_{k,t}$ - stopping time st

$$\tilde{M}_k := W_{T_k} \qquad (\tilde{M}_k)_{k\ge 1} \stackrel{d}{=} (M_k)_{k\ge 1}.$$

- $\tau_k := T_k - T_{k-1}$ $\quad \mathcal{H}_k := \mathcal{G}_{k,T_k}$.

$$\mathbb{E}(\tau_k \mid \mathcal{H}_{k-1}) = \mathbb{E}[\tilde{D}_k^2 \mid \tilde{M}_0, \dots \tilde{M}_{k-1}]$$

$$\mathbb{E}(\tau_k^2 \mid \mathcal{H}_{k-1}) \le C \cdot \mathbb{E}[\tilde{D}_k^4 \mid \tilde{M}_0 \dots \tilde{M}_{k-1}]$$

$$\hat{S}_n(t) = M_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor) D_{\lfloor nt\rfloor+1} \xrightarrow{d} W$$

suff to show $\quad \tilde{S}_n \to W$

$$\tilde{S}_n(t) = \underbrace{W(T_{\lfloor nt\rfloor})}_{\tilde{M}_{\lfloor nt\rfloor}} + \big(\underbrace{W(T_{\lfloor nt\rfloor+1}) - W(T_{\lfloor nt\rfloor})}_{\tilde{D}_{\lfloor nt\rfloor+1}}\big)(nt - \lfloor nt\rfloor)$$

Using the lemma, suff to show $\quad T_{\lfloor nt\rfloor} \xrightarrow{p} t \quad$ (LLN)

$$T_k = \sum_{\ell=1}^{k} \tau_\ell \qquad \tau_\ell = T_\ell - T_{\ell-1}$$

$$\hat{T}_k = \sum_{\ell=1}^{k} \hat{\tau}_\ell \qquad \hat{\tau}_\ell := \tau_\ell - \mathbb{E}(\tau_\ell \mid \mathcal{H}_{\ell-1})$$

$$\mathcal{H}_k = \mathcal{G}_{k, T_k}.$$

$$T_k = \hat{T}_k + \sum_{\ell=1}^{k} \mathbb{E}(\tau_\ell \mid \mathcal{H}_{\ell-1})$$

$$= \hat{T}_k + \sum_{\ell=1}^{k} \mathbb{E}(\tilde{D}_\ell^2 \mid \tilde{M}_0 \dots \tilde{M}_{\ell-1})$$

$$= \hat{T}_k + \langle \tilde{M} \rangle_k \qquad ; \quad \mathbb{E}\hat{T}_k = 0.$$

by assumption $\langle \tilde{M} \rangle_{\lfloor nt \rfloor} \xrightarrow{p} t$; need to show $\hat{T}_{\lfloor nt \rfloor} \xrightarrow{p} 0$

$$\mathrm{Var}(\hat{T}_{\lfloor nt \rfloor}) = \sum_{\ell=1}^{\lfloor nt \rfloor} \mathbb{E}(\hat{\tau}_\ell^2) \leq \sum_{\ell=1}^{\lfloor nt \rfloor} \mathbb{E}(\tau_\ell^2)$$

$\mathbb{E}[\mathbb{E}(\tau_\ell^2 \mid \tilde{M}_0 \dots \tilde{M}_{\ell-1})]$

$$\leq C \sum_{\ell=1}^{\lfloor nt \rfloor} \underline{\mathbb{E}[\mathbb{E}(D_\ell^4 \mid M_0 \dots M_{\ell-1})]}$$

by $|D_k^{(n)}| \leq \epsilon_n$

$$\leq C\epsilon_n^2 \sum_{\ell=1}^{\lfloor nt \rfloor} \mathbb{E}(D_\ell^2) = C\epsilon_n^2\, \mathbb{E}(\langle M \rangle_{\lfloor nt \rfloor})$$

$\langle M \rangle_{\lfloor nt \rfloor} \xrightarrow{p} t$ since bdd. $\mathbb{E} \langle M \rangle_{nt} \to t$

$$\mathrm{Var}(\hat{T}_{\lfloor nt \rfloor}) \to 0$$

$$\mathbb{E}(\tau_\ell^2 \mid \tilde{M}_0, \dots \tilde{M}_{\ell-1}) \leq 2\, \mathbb{E}(\tilde{D}_\ell^4 \mid \tilde{M}_0 \dots \tilde{M}_{\ell-1})$$

(cond on $\tilde{M}_0 .. \tilde{M}_{\ell-1}$ look at cond distr of $\tilde{D}_\ell$

$A_\ell\, B_\ell$ as skorokhod

$$\tau_\ell = \inf\{t : W(T_{\ell-1}+t) - W(T_{\ell-1}) \notin (-A_\ell, B_\ell)\}$$

$$(A_\ell, B_\ell) = \mathcal{F}(U_{2\ell-1}, U_\ell ; \mathbb{P}_{\tilde{D}_\ell | \tilde{M}_0 .. \tilde{M}_{\ell-1}})$$

$$\mathbb{E}(\tau_\ell^2 | \mathcal{H}_{\ell-1})$$

$$\mathbb{E}(\tau_\ell^2) \leq 2\,\mathbb{E}(D_\ell^4)$$

$\blacksquare$

<u>Std construction</u> $\quad (M_k, \mathcal{F}_k)$ a fixed $\underset{L^2}{\wedge}$ MG $\quad M_0 = 0$

$$\left(M_k^{(n)} = \frac{1}{\sqrt{n}} M_k . \right)$$

① $\quad \frac{1}{n} \langle M \rangle_n := \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(D_k^2 | M_0 .. M_{k-1}) \xrightarrow{P} 1$

② $\quad \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[D_k^2 1_{|D_k| \geq \epsilon\sqrt{n}} | M_0, .. M_{k-1}] \xrightarrow{P} 0$

$$\hat{S}_n(t) = \frac{1}{\sqrt{n}} \{ M_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) D_{\lfloor nt \rfloor + 1} \}$$

$$\hat{S}_n \xrightarrow{d} W .$$

$\blacksquare$

<u>Thm</u> (Khinchin) $\qquad h(t) := \sqrt{2t \log\log 1/t}$

$\qquad\qquad\qquad\qquad\quad \tilde{h}(t) := \sqrt{2t \log\log t}$

Then, as

(1) $\limsup_{t\to 0} \frac{W_t}{h(t)} = 1.$ $\qquad$ (2) $\liminf_{t\to 0} \frac{W_t}{h(t)} = -1$

(3) $\limsup_{t\to\infty} \frac{W_t}{\tilde{h}(t)} = 1$ $\qquad$ (4) $\liminf_{t\to\infty} \frac{W_t}{\tilde{h}(t)} = -1.$

$\square$

[(1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) by space inversion $(-W \overset{d}{=} W)$

$(1) \Leftrightarrow (3)$, $(2) \Leftrightarrow (4)$ by time inv $\left((t W_{1/t}) \overset{d}{=} (W_t) \right)$

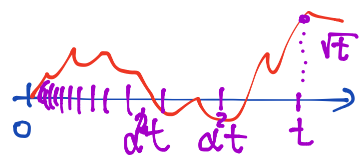
(in particular LIL $\Rightarrow$ W not $1/2$-Hölder).

<u>Proof</u> $\quad \limsup_{t \to 0} \frac{W_t}{h(t)} = 1.$

<u>Upper bound</u> $\quad X_t := e^{\theta W_t - \theta^2 t/2} \quad$ MG $\quad X_0 = 1$

$$\mathbb{P}\left(\sup_{s \in [0,t]} \left(W_s - \frac{\theta s}{2}\right) > y \right) = \mathbb{P}\left(\sup_{s \in [0,t]} X_s > e^{\theta y} \right)$$

$$\leq e^{-\theta y}\, \mathbb{E} X_t = e^{-\theta y}$$



$t_n = \alpha^{2n} \qquad y_n = \frac{h(t_n)}{2}. \quad \alpha \in (0,1), \ \delta > 0$

$\theta_n = (1+2\delta) \frac{h(t_n)}{t_n}. \qquad \mathbb{P}\left(\sup_{s \in [0,t_n]} (W_s - s\theta_n) > y_n \right)$

$$e^{-\theta_n y_n} = e^{-(1+2\delta) \frac{h(t_n)^2}{2 t_n}} = e^{-(1+2\delta) \log\log 1/t_n}$$

$$= \left(\log 1/t_n \right)^{-(1+2\delta)} = \left(n \cdot \log 1/\alpha^2 \right)^{-(1+2\delta)} = C(\alpha,\delta) \cdot n^{-(1+2\delta)}$$

summable! By BC1. $\exists\, N_0 = N_0(\omega; \delta, \alpha)$

st $\forall\, n \geq N_0 \qquad \sup_{t \in [0,t_n]} \left(W_t - t\theta_n/2 \right) \leq y_n$

$$t \in [t_{n+1}, t_n] \Rightarrow W_t \leq \frac{t\theta_n}{2} + y_n \leq \frac{h(t_n)}{2} + \frac{h(t_n)}{2} \frac{(1+2\delta)t}{t_n}$$

$$\frac{W_t}{h(t)} \leq \frac{h(t_n)}{h(t)} \left[\frac{1}{2} + \frac{1}{2}(1+2\delta) \cdot \frac{t}{t_n} \right] \leq \frac{h(t_n)}{h(t_{n+1})} \left[\frac{1}{2} + \frac{1}{2}(1+2\delta) \right]$$

$$\leq \frac{1}{\alpha}(1+\delta)$$

We proved $\forall\ n \geq N_0\ \forall\ t \in [t_{n+1}, t_n]$

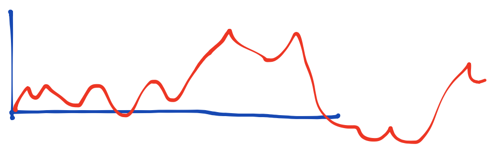
$$\frac{W_t}{h(t)} \leq \frac{1}{\alpha}(1+\delta)$$

$$\forall\ t < \alpha^{2N_0} = t_0(\omega) \qquad \frac{W_t}{h(t)} \leq \frac{1}{\alpha}(1+\delta)$$

$$\mathbb{P}\Big[\underbrace{\limsup_{t\to 0} \frac{W_t}{h(t)} \leq \frac{1}{\alpha_k}(1+\delta_k)}_{\mathcal{E}_k}\Big] = 1$$

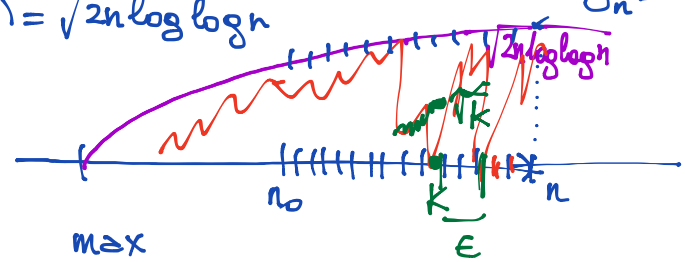
$$\delta_k \downarrow 0 \qquad \alpha_k \uparrow 1$$

$$\mathbb{P}\Big(\bigcap_k \mathcal{E}_k\Big) = \mathbb{P}\Big(\limsup_{t\to 0} \frac{W_t}{h(t)} \leq 1\Big) = 1 .$$



$$\limsup_{n\to\infty} \frac{S_n}{\tilde{h}(n)} = 1$$

$$\tilde{h}(n) = \sqrt{2n\log\log n} \qquad S_n = \sum_{e=1}^{n} \xi_e$$



$$\frac{S_n}{\sqrt{n}} \approx G_n \qquad \mathbb{P}\Big(\max_{n_0 \leq k \leq n} \frac{S_k}{\sqrt{k}} \geq x_n\Big) \lessapprox (n-n_0)\,\mathbb{P}(G_k \geq x_n)$$

$$\log n$$

$$\leq n\, e^{-x_n^2/2} \downarrow 0$$

$$x_n = \sqrt{2\log n}$$

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} \leq 1 \qquad x_n \sim \sqrt{\cancel{2n}\log\log n}\,?$$

$$\underbrace{n,\ n\alpha^2,\ n\alpha^4,\ \dots\ \alpha^{2k} n}\quad \dots \quad \underbrace{n\alpha^{2M}}_{O(1)}$$

$$M \simeq \log n.$$