Lecture 16

We saw that $\forall t \in \mathbb{R}_{\geq 0}$

$$\limsup_{h \downarrow 0} \frac{W_{t+h} - W_t}{\sqrt{2h \log|\log h|}} = +1 \qquad \liminf_{h \downarrow 0} \frac{W_{t+h} - W_t}{\sqrt{2h \log|\log h|}} = -1$$

W is not differentiable at t.

Are there special (random) times t st
W diff at t?

$$D^1 f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}$$

$$D_1 f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}$$

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Thm (Paley-Wiener-Zygmund)

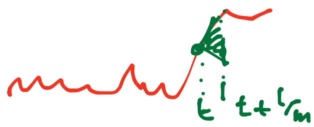
$$\mathbb{P}(\exists t \geq 0 : -\infty < D_1 W_t \leq D^1 W_t < \infty) = 0 \qquad \blacksquare$$

Proof Prove that W nowhere diff in $[0,1]$

$\forall K_0 \in \mathbb{N}$

$$\mathbb{P}(\underbrace{\exists t \in [0,1] \text{ st } -K_0 \leq D_1 W_t < D^1 W_t \leq K_0}_{E_{K_0}}) = 0$$

$$E_{K_0} \subseteq \bigcup_{m=1}^{\infty} \underbrace{\bigcup_{t \in [0,1]} \bigcap_{h \in [0, 1/m]} \{ |W_{t+h} - W_t| \leq K h \}}_{A_{m,k}} \qquad K = K_0 + 1$$



$E_{k_0} \subseteq \bigcup_{m=1}^{\infty} A_{m,k}$

sufficient to show $\mathbb{P}(A_{m,k}) = 0 \quad \forall m,k \in \mathbb{N}$.

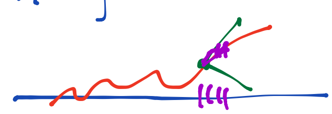
We will show $A_{m,k} \subseteq C \quad \mathbb{P}(C) = 0$

$$A_{m,k} = \bigcup_{t\in[0,1]} \bigcap_{h\in[0,1/m]} \{ |W_{t+h} - W_t| \le Kh \}.$$

Naive discretization

$$C_n = \bigcup_{\ell=0}^{n-1} \left\{ \left|W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}\right| \le \frac{3K}{n} \right\}$$

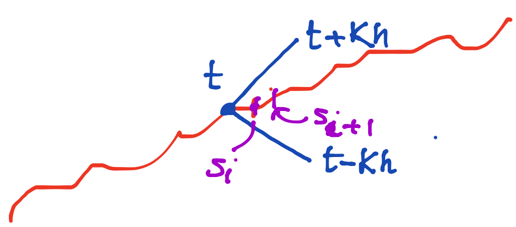
$$C = \bigcap_{n=m}^{\infty} C_n .$$



$A_{m,k} \subseteq C.$

$s_{i+1} - s_i = \frac{1}{n}$

$s_i - t \le 1/n$



$s_i = \frac{i}{n}$

$$|W_{s_{i+1}} - W_{s_i}| \le |W_{s_{i+1}} - W_t| + |W_{s_i} - W_t| \le$$

$$\le K|s_{i+1} - t| + K|s_i - t| \le \frac{2K}{n} + \frac{K}{n} \le \frac{3K}{n}$$

$$\mathbb{P}(C) \le \limsup_{n\to\infty} \mathbb{P}(C_n) \le \limsup_{n\to\infty} \sum_{\ell=0}^{n-1} \mathbb{P}\left(\left|W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}\right| \le \frac{3K}{n}\right)$$

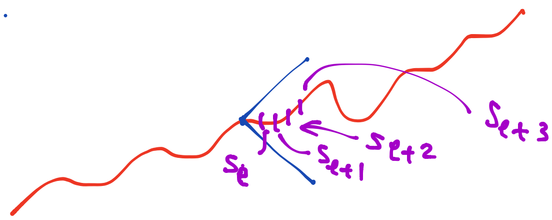
$$\le \limsup_{n\to\infty} n \cdot \mathbb{P}\left(\frac{1}{\cancel{\sqrt{n}}}|G| \le \frac{3K}{\cancel{n}\sqrt{n}}\right) \le \limsup_{n\to\infty} \overset{\sqrt{n}}{\cancel{n}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{6K}{\cancel{\sqrt{n}}} \to \infty$$

$$\mathbb{E}(X_t X_s) = g(t \wedge s)$$

2nd take

$$C_n := \bigcup_{\ell=0}^{n-3} \left\{ \left|W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}\right| \leq \frac{10K}{n},\ \left|W_{\frac{\ell+2}{n}} - W_{\frac{\ell+1}{n}}\right| \leq \frac{10K}{n}, \right.$$
$$\left. \left|W_{\frac{\ell+3}{n}} - W_{\frac{\ell+2}{n}}\right| \leq \frac{10K}{n} \right\}.$$

$$C = \bigcap_{n=30m}^{\infty} C_n.$$



$$\underbrace{|W_{t+h} - W_t| \leq Kh}_{\forall\ h\in[0,\frac{1}{m}]} \implies |W_{s_{i+1}} - W_{s_i}| \leq \frac{10K}{n} \quad i\in\{\ell,\ell+1,\ell+2\}$$

$h\in[0,\frac{3}{n}]$ $\quad n \geq 30$

$$\mathbb{P}(C) \leq \limsup_{n\to\infty} \mathbb{P}(C_n) \leq \limsup_{n\to\infty} \sum_{\ell=0}^{n-3} \mathbb{P}\left(\left|W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}\right| \leq \frac{10K}{n}, \dots\right)$$

$$\leq \limsup_{n\to\infty}\ n\cdot \mathbb{P}\left(\frac{1}{\sqrt{n}}|G_1| \leq \frac{10K}{n}, \frac{1}{\sqrt{n}}|G_2| \leq \frac{10K}{n}, \frac{1}{\sqrt{n}}|G_3| \leq \frac{10K}{n}\right)$$

$$\leq \limsup_{n\to\infty}\ n\cdot\left(\frac{1}{\sqrt{2\pi}}\cdot\frac{20K}{\sqrt{n}}\right)^3 = 0.$$

What is modulus of continuity?

$$Q_{T,\delta}(x) = \sup_{0\le t\le T-\delta} \sup_{0\le h\le\delta} |x(t+h)-x(t)|$$

Notice For a fixed t a.s.

$$\sup_{0\le h\le\delta} |W_{t+h}-W_t| \le \sqrt{2\delta\log\log 1/\delta}\,(1+o_h(1))$$

<u>Thm</u> (Levy) $g(\delta) := \sqrt{2\delta\log 1/\delta}$ Then $\forall\, T<\infty$ a.s.

$$\limsup_{\delta\to 0} \frac{Q_{T,\delta}(W)}{g(\delta)} = 1. \qquad \square$$

<u>Rmk</u> $Q_{\infty,\delta}(W) = \infty \quad \forall\, \delta>0$.

Indeed $Q_{\infty,\delta}(W) \ge \sup_{k\in\mathbb{N}} |W_{(k+1)\delta} - W_{k\delta}|$

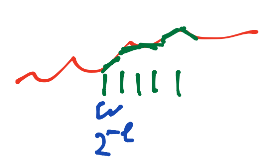
$\stackrel{\cdot}{=} \sqrt{\delta} \sup_{k\in\mathbb{N}} |G_k| = \infty. \qquad (G_k)_{k\in\mathbb{N}}$ iid $N(0,1)$

<u>Proof</u> By scaling $T=1$ $\quad Q_{1,\delta} = Q_\delta$.

<u>Lower bound</u>

$$\Delta_\ell(x) = \max_{0\le k\le 2^\ell-1} \left| x\left(\frac{k+1}{2^\ell}\right) - x\left(\frac{k}{2^\ell}\right)\right|$$

$$Q_{2^{\ell}}(x) \ge \Delta_\ell(x)$$



$$\mathbb{P}(\Delta_\ell(W) \geq u) = 1 - \mathbb{P}(|W_{2^{-\ell}}| < u)^{2^\ell}$$

$$= 1 - \left(1 - \mathbb{P}(|G| \geq u 2^{\ell/2})\right)^{2^\ell}$$

$$\geq 1 - \exp\left(-2^\ell \mathbb{P}(|G| \geq u 2^{\ell/2})\right)$$

$$\mathbb{P}(|G| \geq x) \geq C e^{-(1+\xi) x^2/2} \qquad \xi > 0$$

$$\geq 1 - \exp\left\{-C 2^\ell \cdot e^{-(1+\xi) u^2 2^\ell / 2}\right\}$$

$$u_\ell = \frac{1-\epsilon}{\sqrt{1+\xi}}\, g(2^{-\ell}) = \frac{1-\epsilon}{\sqrt{1+\xi}} \sqrt{2 \cdot 2^{-\ell} \log 2^\ell}$$

$$\mathbb{P}(\Delta_\ell(W) \geq u_\ell) \geq 1 - \exp\left\{-C 2^\ell e^{-(1-\epsilon)^2 \log 2^\ell}\right\}$$

$$= 1 - \exp\left(-C 2^{\ell \overbrace{[1-(1-\epsilon)^2]}^{>0}}\right)$$

$$\sum_{\ell=1}^\infty \mathbb{P}(\Delta_\ell(W) < u_\ell) < \infty$$

By BC1 $\exists\, \ell_0 = \ell_0(\omega, \epsilon)$ st $\forall \ell \geq \ell_0$

$$\Delta_\ell(W) \geq u_\ell = g(2^{-\ell}) \cdot (1-\tilde{\epsilon})$$

$$Q_{2^{-\ell}}(W) \geq (1-\tilde{\epsilon})\, g(2^{-\ell}) \qquad \forall \ell \geq \ell_0$$

$$\limsup_{\ell \to \infty} \frac{Q_{2^{-\ell}}(W)}{g(2^{-\ell})} \geq 1 - \tilde{\epsilon}$$

$$\limsup_{\delta \downarrow 0} \frac{Q_\delta(W)}{g(\delta)} \geq 1 - \tilde{\epsilon}.$$

Upper bound

$$\Delta_{\ell,m}(x) = \max_{0 \le k \le 2^{\ell}-m} \left| x\left(\frac{k+m}{2^{\ell}}\right) - x\left(\frac{k}{2^{\ell}}\right) \right|$$

Lemma Assume $\exists\, \eta \in (0,1)$ $\ell_0 \ge 1$ st $\forall \ell \ge \ell_0$
$\forall\, m \in \{1, \dots, 2^{\ell\eta}\}$

$$\Delta_{\ell,m}(x) \le \sqrt{b} \cdot g(m \cdot 2^{-\ell})$$

Then if x cont. $\exists\, \epsilon(h) \downarrow 0$ as $h \downarrow 0$ st.

$$\sup_{0 \le t \le 1-h} \underline{|x(t+h) - x(t)| \le \sqrt{b}\, g(h)\,(1 + \epsilon(h))} \qquad \blacksquare$$

Need to prove a.s. $\forall b > 1$ $\Delta_{\ell,m}(W) \le \sqrt{b}\, g(m \cdot 2^{-\ell})$ $\forall \ell \ge \ell_0$, $m \le 2^{\ell\eta}$

Fix η, b

$$B_\ell := \left\{ \exists\, m \in \{1, 2, \dots, 2^{\eta\ell}\} \text{ st } \Delta_{\ell,m}(W) > \sqrt{b}\, g(m 2^{-\ell}) \right\}$$

$$\mathbb{P}(B_\ell) \le \sum_{m=1}^{2^{\eta\ell}} \mathbb{P}\left(\Delta_{\ell,m}(W) > \sqrt{b}\, g(m 2^{-\ell})\right)$$

$$\le \sum_{m=1}^{2^{\eta\ell}} \sum_{k=0}^{2^{\ell}-m} \mathbb{P}\left(\left| W_{\frac{k+m}{2^{\ell}}} - W_{\frac{k}{2^{\ell}}} \right| > \sqrt{b}\, g(2^{-\ell} m) \right)$$

$$\le 2^{\ell} \sum_{m=1}^{2^{\eta\ell}} \mathbb{P}\left(\sqrt{\frac{\cancel{m}}{\cancel{2^{\ell}}}}\, |G| \ge \sqrt{b} \sqrt{2 \cdot \cancel{2^{-\ell} m} \log \frac{2^{\ell}}{m}} \right)$$

$$\le 2^{\ell} \sum_{m=1}^{2^{\ell\eta}} \mathbb{P}\left(|G| \ge \sqrt{2b \log 2^{\ell}/m} \right)$$

$$\mathbb{P}(|G| \geq t) \leq C e^{-t^2/2}$$

$$\leq C 2^{\ell} \sum_{m=1}^{2^{\eta \ell}} \exp\left\{ -\frac{1}{\cancel{2}} \cancel{2} b \log \frac{2^{\ell}}{m} \right\}$$

$$\leq C 2^{\ell(1-b)} \sum_{m=1}^{2^{\eta \ell}} m^{b} \qquad b > 1$$

$$\left[\sum_{m=1}^{M} m^{b} \leq C M^{b+1} \right]$$

$$\mathbb{P}(B_{\ell}) \leq C 2^{\ell(1-b)} \cdot 2^{\eta \ell (1+b)} = C 2^{\ell \overbrace{[1-b+\eta(1+b)]}^{q(\eta)}}$$

$$\eta = 0 \implies q(0) = 1 - b < 0$$

$\forall\, b > 1 \ \exists\, \eta > 0$ st $q(\eta) < 0$ choose such η.

$$\sum_{\ell=1}^{\infty} \mathbb{P}(B_{\ell}) \leq C \sum_{\ell=1}^{\infty} 2^{q(\eta)\ell} < \infty .$$

$$B_{\ell} := \left\{ \exists\, m \in \{1, .. 2^{\eta \ell}\} \text{ st } \Delta_{\ell,m}(W) \geq \sqrt{b}\, g(2^{-\ell} m) \right\}$$

by BCI $\exists\, \ell_0 = \ell_0(\omega)$ st $\forall\, \ell \geq \ell_0$

$\forall\, m \in \{1, .. 2^{\eta \ell}\}$ $\quad \Delta_{\ell,m}(W) < \sqrt{b}\, g(2^{-\ell} m)$.

$$\limsup_{\delta \to 0} \frac{1}{g(\delta)} \cdot \sup_{0 \leq t \leq 1-\delta} |W_{t+\delta} - W_t| \leq \sqrt{b}$$

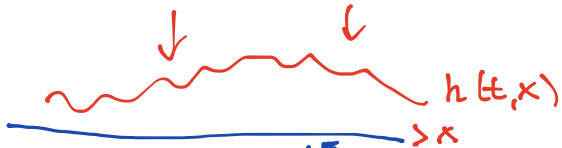
$$\implies \lim_{\delta \to 0} \frac{1}{g(\delta)} \sup_{0 \leq h \leq \delta} \sup_{0 \leq t \leq 1-h} |W_{t+h} - W_t| \leq \sqrt{b} .$$

$$\lim_{\delta \to 0} \frac{1}{g(\delta)} Q_{\delta}(W) \leq \sqrt{b} \qquad b_k = 1 + \frac{1}{k} \downarrow 1$$

1+1 dim $\qquad h(t,x)$

$$\partial_t h = \partial_{xx} h + (\partial_x h)^2 + \eta(t,x)$$

KPZ.



$$\mathbb{E}\, \eta(t,x)\, \eta(t',x') = \delta(t-t')\, \delta(x-x')$$

$$B_t = \int_0^t \eta(t')\, dt'.$$

$$\ell(h) = \sqrt{2h \log\log 1/h}$$

$$\limsup_{h \downarrow 0} \frac{W_{t+h} - W_t}{\ell(h)} = 1$$

$$\limsup_{h \to 0} \frac{W_{t+h} - W_t}{h} = \limsup_{h \to 0} \overbrace{\frac{\ell(h)}{h}}^{\infty}\ \overbrace{\frac{W_{t+h} - W_t}{\ell(h)}}^{1} = \infty.$$