Lecture 18 , 5/27/2021

Thm $(H_t)_{t\geq 0}$ adapted progressive $\int_0^t \mathbb{E} H_s^2 \, ds < \infty$ $\forall t$. Let

$$M_t := \int_0^t H_s \, dB_s$$

Then $(M_t)_{t\geq 0}$ admits a cont modification which is a MG with $\mathbb{E} M_t = 0$.

Proof Fix t_* large time will prove cont of $(M_t)_{t\leq t_*}$. $\{H_n\}$ step process $\|H_n - H^{t_*}\|_2 \to 0$

We know

$$\left\| \int H_n \, dB - \int H^{t_*} dB \right\|_{L^2} \to 0$$

$$X(t) = \mathbb{E}\left\{ \int_0^{t_*} H(s) \, dB(s) \,\Big|\, \mathcal{F}_t \right\}$$

This is a MG.

Fix n. $H_n = \sum_i A_i \mathbb{1}_{(t_i, t_{i+1}]}$ $s < t$, $t_i \leq s < t_{i+1}$, $t_j \leq t < t_{j+1}$

wlog $i < j$

$$\int_0^t H_n(u) \, du = \overbrace{\sum_{\ell=0}^{i-1} A_\ell (B_{t_{\ell+1}} - B_{t_\ell}) + A_i (B_s - B_{t_i})}^{\int_0^s H_n \, dB}$$

$$+ \underbrace{A_i (B_{t_{i+1}} - B_s) + \sum_{\ell=i+1}^{j-1} A_\ell (B_{t_{\ell+1}} - B_{t_\ell}) + A_j (B_t - B_{t_j})}_{\mathbb{E}(\cdot \mid \mathcal{F}_s) = 0}$$

$$\mathbb{E}\left\{ \int_0^t H_n(u)\,dB(u) \,\Big|\, \mathcal{F}_s \right\} = \int_0^s H_n(u)\,dB(u)$$

Hence $\left(\int_0^t H_n(u)\,dB(u) \right)_{u \geq 0}$ is a MG

$$\circledast\ \mathbb{E}\left\{ \sup_{t \leq t_*} \left(X(t) - \int_0^t H_n(s)\,dB(s) \right)^2 \right\} \leq \quad \text{(Doob)}$$

$$\leq C\ \mathbb{E}\left\{ \left(X(t_*) - \int_0^{t_*} H_n(u)\,dB(u) \right)^2 \right\}$$

$$\left[\int_0^{t_*} H(u)\,dB(u) \right.$$

$$= C\ \| H^{t_*} - H_n \|_2^2 \longrightarrow 0$$

Can take a subseq st $\sum_n \| H^{t_*} - H_n \|_2 < \infty$

$$\Rightarrow \sup_{t \leq t_*} \Big| X(t) - \underbrace{\int_0^t H_n(s)\,dB(s)}_{\text{cont}} \Big| \to 0 \quad \text{a.s.}$$

$\Rightarrow (X(t))_{t \geq 0}$ cont. MG $\quad \mathbb{E}X(t)$

$$\int_0^t H_n(s)\,dB(s) \longrightarrow \int_0^t H(s)\,dB(s) \quad \text{a.s.}$$

$$\Rightarrow \quad X(t) = \int_0^t H(s)\,dB(s).$$

$\square$

$$\left[\quad X(t) := \mathbb{E}\left\{ \int_0^{t_*} H(s)\,dB(s) \,\Big|\, \mathcal{F}_t \right\} \quad \right]$$

Itô's formula.

[Calculus : $f(x(t)) - f(x(0)) = \int_0^t f'(x(s))\, dx(s)$.]

Thm $(Z(t))_{t\geq 0}$ increas cont adapted. wrt $\mathcal{F}_t$

$(\mathcal{F}_t, B(t))_{t\geq 0}$ Wiener. $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

twice cont diff wrt to first argument

once " " " " second "

Assume
$$\int_0^{t_0} \mathbb{E}\{\partial_x f(B(t), Z(t))^2\}\, dt < 0$$

Then a.s : for all $0 \leq t \leq t_0$

$$f(B(t), Z(t)) - f(B(0), Z(0)) = \int_0^t \partial_x f(B(s), Z(s))\, dB(s)$$
$$+ \int_0^t \partial_y f(B(s), Z(s))\, dZ(s) + \frac{1}{2}\int_0^t \partial_{xx} f(B(s), Z(s))\, ds .$$

[" $df(B(t)) = f'(B(t))\, dB(t) + \frac{1}{2} f''(B(t))\, dt$ "]

Lemma $f:\mathbb{R}\to\mathbb{R}$ cont $0=t_0^{(n)}<t_1^{(n)}<\dots<t_n^{(n)}=t$

$$\max_{j\le n-1}|t_{j+1}^{(n)}-t_j^{(n)}|=:\delta_n\downarrow 0$$

$$\operatorname{plim}_{n\to\infty}\sum_{j=0}^{n-1}f(B(t_j^{(n)}))\,(B(t_{j+1}^{(n)})-B(t_j^{(n)}))^2=\int_0^t f(B(s))\,ds$$

Proof $T:=\inf\{t\ge 0:\ B(t)\notin(-M,M)\}$

$$B_M(t)=B(t\wedge T).$$

Will prove

$$\operatorname{plim}_{n\to\infty}\left|\sum_{\ell=0}^{n-1}f(B_M(t_\ell))\,(B_M(t_{\ell+1})-B_M(t_\ell))^2-\int_0^{t\wedge T}f(B(s))\,ds\right|=0 \quad (*)$$

$\forall\epsilon>0$ can take M so that $\mathbb{P}(T\le t)\le\epsilon$.

By cont of $B_M(t)$, a.s.

$$\sum_{\ell=0}^{n-1}f(B_M(t_\ell))\,(t_{\ell+1}\wedge T-t_\ell\wedge T)\to\int_0^{T\wedge t}f(B(s))\,ds.$$

$(**)$

sufficient to show

$$\Big\|\sum_{\ell=0}^{n-1}\underbrace{f(B_M(t_\ell))}_{F_\ell}\Big[\underbrace{(B_M(t_{\ell+1})-B_M(t_\ell))^2-(t_{\ell+1}\wedge T-t_\ell\wedge T)}_{Z_{\ell+1}}\Big]\Big\|_{L^2}\to 0$$

Note that $\overbrace{\mathbb{E}[Z_{\ell+1}\,|\,\mathcal{H}_\ell]=0}^{(*)}$ $\qquad\mathcal{H}_\ell=\mathcal{F}_{T\wedge t_\ell}.$

$$\Big\|\sum_{\ell=0}^{n-1}F_\ell Z_{\ell+1}\Big\|_{L^2}^2=\sum_{\ell=0}^{n-1}\|F_\ell Z_{\ell+1}\|_{L^2}^2+2\sum_{\ell_1<\ell_2}\underbrace{\mathbb{E}[F_{\ell_1}Z_{\ell_1+1}F_{\ell_2}Z_{\ell_2+1}]}_{0}$$

$$= \sum_{\ell=0}^{n-1} \| F_\ell Z_{\ell+1} \|_{L^2}^2 \qquad |F_\ell| \le \sup_{|x|<M} |f(x)| =: C$$

$$\le C \sum_{\ell=0}^{n-1} \| Z_{\ell+1} \|_{L^2}^2$$

$$\le C \sum_{\ell=0}^{n-1} \mathbb{E}\{ (B_M(t_{\ell+1}) - B_M(t_\ell))^4 \}$$

$$\le C' \sum_{\ell=0}^{n-1} (t_{\ell+1} - t_\ell)^2 \le C' \overbrace{\sup_\ell |t_{\ell+1} - t_\ell|}^{\delta_n} \cdot t \to 0$$

(*) to show $\mathbb{E}[Z_{\ell+1} | \mathcal{F}_{T \wedge t_\ell}] = 0$

that is

$$\mathbb{E}[(B(t_{\ell+1} \wedge T) - B(t_\ell \wedge T))^2 | \mathcal{F}_{t_\ell \wedge T}] = \mathbb{E}\{(t_{\ell+1} \wedge T - t_\ell \wedge T) | \mathcal{F}_{T \wedge t_\ell}]$$

that is

$$\mathbb{E}[B(t_{\ell+1} \wedge T)^2 - t_{\ell+1} \wedge T | \mathcal{F}_{t_\ell \wedge T}] = B(t_\ell \wedge T) - t_\ell \wedge T.$$

true because $X_t = (B(t)^2 - t)$ is MG $\Rightarrow$ $X_{t \wedge T}$ is MG

<u>Proof of Ito</u>: Partitions $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$.

$$\max_\ell |t_{\ell+1}^{(n)} - t_\ell^{(n)}| =: \delta_n \to 0.$$

$$\operatorname{plim}_{n \to \infty} \sum_{\ell=0}^{n-1} \partial_{xx} f(B(t_\ell), Z(t_\ell)) (B(t_{\ell+1}) - B(t_\ell))^2 =$$

$$= \int_0^t \partial_{xx} f(B(s), \underbrace{Z(s)}) \, ds.$$

Want to compute

$$f(B(t),Z(t)) - f(B(0),Z(0)) = \underbrace{\sum_{\ell=0}^{n-1}\left[f(B(t_{\ell+1}),Z(t_{\ell+1})) - f(B(t_\ell),Z(t_\ell))\right]}$$

$$Q^{(1)}_{\delta,M}(f) = \sup_{\substack{|x_i|,|y_i|\le M \\ |x_1-x_2|\le\delta \\ |y_1-y_2|\le\delta}} |\partial_y f(x_1,y_1) - \partial_y f(x_2,y_2)|$$

$$Q^{(2)}_{\delta,M}(f) = \sup_{\substack{|x_i|\ |y_i|\le M \\ |x_1-x_2|\le\delta,\ |y_1-y_2|\le\delta}} |\partial_{xx} f(x_1,y_1) - \partial_{xx} f(x_2,y_2)|$$

By assumption $Q^{(1/2)}_{\delta,M} \downarrow 0$ as $\delta \downarrow 0$

Taylor $\forall (x_0,y_0),\ (x,y) \in [-M,M]^2 \quad |x-x_0|,\ |y-y_0| \le \delta$

$$\Big| f(x,y) - f(x_0,y_0) - (x-x_0)\partial_x f(x_0,y_0) - \frac{1}{2}(x-x_0)^2 \partial_{xx} f(x_0,y_0) - (y-y_0)\partial_y f(x_0,y_0)\Big|$$

$$\le Q^{(1)}_{M,\delta}(f)\cdot |y-y_0| + Q^{(2)}_{\delta,M}(f)\ |x-x_0|^2$$

$$\Big| f(B(t),Z(t)) - f(B(0),Z(0)) - \sum_{\ell=0}^{n-1} \overbrace{\partial_x f(B_{t_\ell},Z_{t_\ell})(B_{t_{\ell+1}} - B_{t_\ell})}^{\to \int \partial_x f\, dB}$$

$$- \underbrace{\frac{1}{2}\sum_{\ell=0}^{n-1} \partial_{xx} f(B_{t_\ell},Z_{t_\ell})(B_{t_{\ell+1}}-B_{t_\ell})^2}_{\int \partial_{xx} f\, dt} - \underbrace{\sum_{\ell=0}^{n-1} \partial_y f(B_{t_\ell},Z_{t_\ell})(Z_{t_{\ell+1}}-Z_{t_\ell})}_{\int \partial_y f\, dZ}\Big|$$

$$\le Q^{(1)}_{\delta_n,M}(f)\cdot \sum_{\ell=0}^{n-1} |Z_{t_{\ell+1}} - Z_{t_\ell}| + Q^{(2)}_{\delta_n,M}(f) \sum_{\ell=0}^{n-1} (B_{t_{\ell+1}} - B_{t_\ell})^2$$

$$M = \max_s \left[|B_s| \vee |Z_s|\right], \quad \delta_n = \max_\ell \left[|B_{t_{\ell+1}} - B_{t_\ell}| \vee |Z_{t_{\ell+1}} - Z_{t_\ell}|\right]$$

$$\sum_{\ell=0}^{n-1} |Z_{t_{\ell+1}} - Z_{t_\ell}| = Z_t - Z_0 \quad , \quad \sum_{\ell=0}^{n-1} (B_{t_{\ell+1}} - B_{t_\ell})^2 \xrightarrow{a.s.} t$$

M indep of n.

RHS $\to 0$

$\delta_n \to 0$ a.s.

For fixed $t \le t_0$ we proved a.s.

$$f(B_t, Z_t) - f(B_0, Z_0) = \text{ITO}(t) \qquad \mathcal{E}_t$$

$$\Rightarrow \quad \mathbb{P}\Big(\bigcap_{t \in [0,t_0] \cap \mathbb{Q}} \mathcal{E}_t \Big) = 1$$

Both LHS and RHS are cont a.s.

$$\Rightarrow \quad \mathbb{P}\Big(\bigcap_{t \in [0,t_0]} \mathcal{E}_t \Big) = 1.$$

$$\underbrace{[B(t+\delta) - B(t)]^2}_{\delta \cdot G^2} = \delta \ \underbrace{\pm\ \delta}$$

$G \sim N(0,1)$

$$\sum_{\ell=0}^{n-1} H\Big(\underbrace{\frac{t_\ell + t_{\ell+1}}{2}}_{\tilde{t}_\ell} \Big) \cdot (B_{t_{\ell+1}} - B_{t_\ell})$$

Z_t increas.

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)\, dZ_s$$

Fix ω $\qquad f(Z_t(\omega)) - f(Z_0(\omega)) = \int_0^t f'(Z_s(\omega))\, dZ_s(\omega)$