Lecture 2

Last lect : given fdd μ $\exists (\Omega, \mathcal{F}, \mathbb{P})$
$(X_t)_{t\in\mathbb{T}}$ whose fdd coincide with μ

Problem $\sup_{t\in\mathbb{T}} X_t$ not meas. and same for other quantities

Example $\mathbb{T}=[0,1]$ $\mu_{t_1\ldots t_n}(A_1\times\ldots\times A_n)=\begin{cases}1 & \text{if } 0\in A_1\ldots A_n\\ 0 & \text{ow}\end{cases}$

$$\Omega=\mathbb{R}^{[0,1]}, \quad \mathcal{F}=\mathcal{B}^{[0,1]}, \quad X_t(\omega)=\omega(t)$$

$$\{\omega : X_t(\omega)\geq -100 \ \forall t\} \notin \mathcal{B}^{[0,1]} .$$

Idea Construct $(Y_t)_{t\in[0,1]}$ st. $\{Y_t\geq 0 \ \forall t\}$ countably represented

Def · $(X_t)_{t\in\mathbb{T}}$ $(Y_t)_{t\in\mathbb{T}}$ versions of one another if they have the same fdd's.

$\Uparrow$ – $(Y_t)_{t\in\mathbb{T}}$ is a modification of $(X_t)_{t\in\mathbb{T}}$ if
$\forall t\in\mathbb{T}$ $\quad \mathbb{P}(X_t\neq Y_t)=0$

$\Uparrow$ – $(X_t),(Y_t)$ are indistinguishable if
$A=\{\omega : X_t(\omega)\neq Y_t(\omega) \text{ for some } t\in\mathbb{T}\}$
A in $\mathbb{P}$-null. $(A\subseteq N \quad \mathbb{P}(N)=0)$

(Typically consider processes up undist.)

$X_t(\omega)=0$ always $\qquad \tilde{X}_t(\omega)=1_{t=\omega}$ $\quad \omega\in[0,1]$, $t\in\mathbb{T}=[0,1]$

Given $(X_t)_{t\in\mathbb{T}}$ want $(Y_t)_{t\in\mathbb{T}}$ modif of X that is continuous $\quad \mathbb{T}=[a,b]^d$

$$\{Y\in C(\mathbb{T})\}\cap\left\{\sup_{t\in\mathbb{T}} Y_t(\omega)\leq\gamma\right\} = \left\{\sup_{t\in\mathbb{T}\cap\mathbb{Q}^d} Y_t(\omega)\leq\gamma\right\}$$
$$\cap\{Y\in C(\mathbb{T})\}$$

Thm (Kolmogorov-Centsov) $(X_t)_{t\in\mathbb{T}}$ $\quad\mathbb{T}=[a,b]^d$

Assume $\exists\ \alpha,\beta,c>0$ st.

$$\mathbb{E}\left[|X_t-X_s|^\alpha\right]\leq c\,\|t-s\|^{\beta+d}\qquad\forall s,t\in\mathbb{T}$$

Then $\exists$ a cont. modif. $(Y_t)_{t\in\mathbb{T}}$ of $(X_t)_{t\in\mathbb{T}}$ measurable on $\mathcal{F}^X$ that is γ-Hölder $\forall\gamma\in(0,\beta/\alpha)$. ($\exists$ finite rv $\tau(\omega)$ st

$$\mathbb{P}\left(\left\{\omega:\sup_{\substack{t,s\in\mathbb{T}\\ t\neq s}}\frac{|Y_t(\omega)-Y_s(\omega)|}{\|t-s\|^\gamma}\leq\tau(\omega)\right\}\right)=1.\ )$$

$\square$

Rmk $\quad \mathbb{E}(|X_t-X_s|^\alpha)\leq c\|t-s\|^{d+\beta}\qquad \gamma<\beta/\alpha$

if determ $\quad |x_t-x_s|^\alpha\leq c\|t-s\|^{\beta+d}$

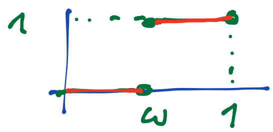
$$|x_t-x_s|\leq c'\,\|t-s\|^{\beta/\alpha+\frac{d}{\alpha}}$$

$$\frac{\beta}{\alpha}+\frac{d}{\alpha}\ \text{Hölder.}$$

We are loosing this reg. bc of randomness.

Example $\quad d=1.\quad \Omega=[0,1]\quad\mathcal{F}=\mathcal{B},\ \mathbb{P}=\mathrm{Unif}([0,1])$

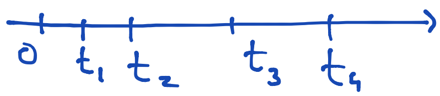
$X_t(\omega) = \mathbb{1}_{t \geq \omega}$



$t > s$

$\mathbb{E}[|X_t - X_s|^\alpha] = |t-s|$ (to get cont would need $\mathbb{E}(|X_t - X_s|^\alpha) \leq C|t-s|^{1+\beta}$, $\beta > 0$)

Right continuous. left limits RCLL.



$\{(t_{i+1} - t_i)\}_{i \geq 0}$ iid Exp(1)

$N_t = \#\{i : t_i \leq t\}$ PPP.

Proof $d = 1$. $\mathbb{T} = [0,1]$

$$\mathbb{P}(|X_t - X_s| \geq \epsilon) \leq \epsilon^{-\alpha} \mathbb{E}[|X_t - X_s|^\alpha] \leq c\, \epsilon^{-\alpha} |t-s|^{\beta+1}$$

$$\mathbb{Q}_1^{(2)} = \bigcup_{\ell=1}^{\infty} \mathbb{Q}_1^{(2,\ell)}$$

$$\mathbb{Q}_1^{(2,\ell)} = \{ j/2^\ell : j \in \{0,1,\dots 2^\ell\}\}$$

Claim $\exists$ rv $\delta(\omega)$ $\mathbb{P}(\delta(\omega) < \infty) = 1$ st

$$\mathbb{P}(|X_t(\omega) - X_s(\omega)| \leq c|t-s|^\gamma \ \forall t,s \in \mathbb{Q}^{(2)} \ |t-s| \leq \delta(\omega))$$

$= 1$.

X_t is a.s. γ-Hölder on diadic rationals

Let us prove the claim. Look at level ℓ

$$\mathbb{P}\Big(\max_{k \in \{0,\dots 2^\ell\}} |X_{(k+1)/2^\ell} - X_{k/2^\ell}| \geq (2^{-\ell})^\gamma\Big) \leq$$

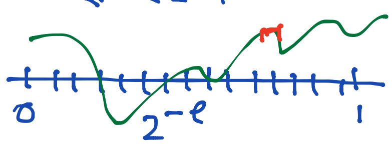
$$\le 2^{\ell}\left(2^{-\ell\gamma}\right)^{-\alpha}\left(2^{-\ell}\right)^{\beta+1} = 2^{-\ell(\beta+\cancel{\gamma}-\alpha\gamma-\cancel{\gamma})} = 2^{-\ell(\beta-\alpha\gamma)}$$

$\gamma < \beta/\alpha \qquad \beta-\alpha\gamma = \eta > 0 \qquad\qquad = 2^{-\ell\eta}$

By BC 1 : $\exists\, N_0(\omega) < \infty$ a.s.

st $\forall\, \ell \geq N_0(\omega)$

$$\max_{0\le k\le 2^{\ell}-1} \left|X_{(k+1)/2^{\ell}} - X_{k/2^{\ell}}\right| \le 2^{-\gamma\ell}$$



$\ell \geq N_0(\omega)$

On this event

$$|X_t - X_s| \le c\,|t-s|^{\gamma} \qquad \forall\ |t-s| \le \underbrace{2^{-N_0(\omega)}}_{\delta(\omega)}$$

$t, s \in \mathbb{Q}_1^{(2)}$ -

$$\sup_{\substack{|t-s|\le 2^{-N_0} \\ t,s\in\mathbb{Q}^{(2)}}} \frac{|X_t - X_s|}{|t-s|^{\gamma}} \le c\,; \quad \mathcal{N} := \{\omega : N_0(\omega) = \infty\}$$

Let's try to construct Y_t γ-Hölder for fixed $\gamma < \beta/\alpha$.

$\omega \in \mathcal{N}$: $\quad Y_t(\omega) = 0 \quad \forall t$

$\omega \notin \mathcal{N} \quad t \in [0,1]$

- $t \in \mathbb{Q}_1^{(2)} \implies Y_t(\omega) := X_t(\omega)$
- $t \notin \mathbb{Q}_1^{(2)}$ fix sequence $(t_k)_{k\in\mathbb{N}}$ $t_k \to t$, $t_k \in \mathbb{Q}_1^{(2)}$ (non. random)

$$Y_t(\omega) := \lim_{k\to\infty} X_{t_k}(\omega)$$

Limit exists because $(X_{t_k}(\omega))_k$ is Cauchy

- Y_t measurable on $\mathcal{F}^X$ (countably repr.)
- Immediate to check (Y_t) γ Holder.

$$|Y_t - Y_s| = \lim_{k\to\infty} |X_{t_k} - X_{s_k}| \leq \lim_{k\to\infty} c|t_k - s_k|^\gamma$$
$$\leq c|t-s|^\gamma .$$

- Is it a modification of (X_t)

$t \in \mathbb{Q}_1^{(2)} \Rightarrow Y_t(\omega) = X_t(\omega) \qquad \forall \omega \notin \mathcal{N}$

$t \notin \mathbb{Q}_1^{(2)}$: $\quad t_k \to t$

$$\mathbb{E}[|X_{t_k}(\omega) - X_t(\omega)|^\alpha] \xrightarrow{k\to\infty} 0$$

$$X_{t_k} \xrightarrow{P} X_t$$

Take a subseq $(\tilde{t}_k) \subseteq (t_k)$

$$\mathbb{P}\big(\underbrace{\lim_{k\to\infty} X_{\tilde{t}_k}(\omega)} = X_t(\omega)\big) = 1$$

$$\mathbb{P}\big(\underbrace{\lim_{k\to\infty} X_{\tilde{t}_k}(\omega)} = Y_t(\omega)\big) = 1$$

$$\Rightarrow \quad \mathbb{P}(X_t(\omega) = Y_t(\omega)) = 1.$$

Concludes the proof for fixed γ.

Want γ-Hölder $\forall \gamma < \beta/\alpha$.

Take a seq $\gamma_k \uparrow \beta/\alpha$, $\gamma_k < \beta/\alpha$.

$\mathcal{N}_k \supseteq \{ X \text{ not } \gamma_k\text{-Hölder on } \mathbb{Q}_1^{(2)} \}$.

$\mathcal{N} := \bigcup_{k=1}^{\infty} \mathcal{N}_k$. and proceed as before. $\blacksquare$

$$\checkmark \quad \sup_{\substack{|t-s|\le\delta \\ t,s\in[0,1]}} \frac{|x(t)-x(s)|}{|t-s|^\gamma} \le C \implies \sup_{t,s\in[0,1]} \frac{|x(t)-x(s)|}{|t-s|^\gamma} \le C'$$

$$\sup_{|t-s|>\delta} \frac{|x(t)-x(s)|}{|t-s|^\gamma} \le 2\delta^{-\gamma} \sup_{t\in[0,1]} |x(t)|$$

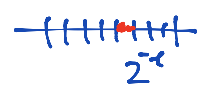
We proved $\exists\, \delta(\omega)$ $\quad \mathbb{P}(\delta(\omega)>0)=1 \quad \bar{C}<\infty$

$$\mathbb{P}\left(\sup_{\substack{t,s\in[0,1] \\ |t-s|\le\delta(\omega)}} \frac{|Y_t(\omega)-Y_s(\omega)|}{|t-s|^\gamma} \le \bar{C}_\gamma \ \forall \gamma < \frac{\beta}{\alpha} \right) = 1$$

$$\mathbb{P}\left(\sup_{t,s\in[0,1]} \frac{|Y_t(\omega)-Y_s(\omega)|}{|t-s|^\gamma} \le c(\omega) \right) = 1$$

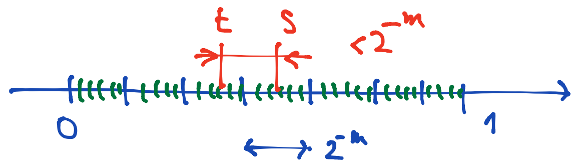
The step we skipped $\quad x: [0,1] \to \mathbb{R}$

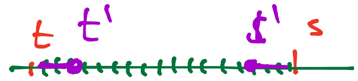
- $\Delta_\ell(x) := \max_{j\le 2^\ell-1} |x(2^{-\ell}(j+1)) - x(2^{-\ell}j)|$



- $\forall k \ge m+1$

$$\max_{\substack{t,s\in\mathbb{Q}^{(2,k)} \\ |t-s|<2^{-m}}} |x(t)-x(s)| \le 2 \sum_{\ell=m+1}^{k} \underline{\Delta_\ell(x)}$$





$$\underset{\nwarrow \mathbb{Q}^{(2,k)}}{|x(t)-x(s)|} \leq |x(t)-x(t')| + |x(s)-x(s')| + \underset{\nwarrow \mathbb{Q}^{(2,k-1)}}{|x(t')-x(s')|}$$

$$\leq \Delta_k(x) + \Delta_k(x) + |x(t')-x(s')|$$

$$|t'-s'| < 2^{-m}$$

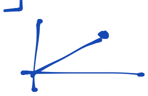
If $\underline{\Delta_\ell(x) \leq 2^{-\gamma\ell}}$ $\underline{\forall \ell > N}$ then for $m \geq N$

$$\max_{\substack{t,s \in \mathbb{Q}^{(2,k)} \\ 2^{-(m+1)} \leq |t-s| < 2^{-m}}} |x(t)-x(s)| \leq 2\sum_{\ell=m+1}^{k} 2^{-\gamma\ell} \leq \underbrace{\left(\frac{2\cdot 2^{-\gamma}}{1-2^{-\gamma}}\right)} 2^{-\gamma m} \leq C\, 2^{-\gamma m}$$

$$\leq C\, |t-s|^{\gamma}.$$

$G(\omega) = G \sim N(0,1)$ $\quad t \in [0,1]$

$$X_t(\omega) = Gt$$



$I_n = [0,n]$

$$\sup_{t,s \leq n} \frac{|X^n_t - X^n_s|}{|t-s|^{\gamma}} \leq C_n(\omega)$$

$$X_t = X^n_t \quad , n-1 \leq t < n$$

$$\sup_{t,s \leq T} \frac{|X_t - X_s|}{|t-s|} \leq \max_{n \leq \lceil T \rceil} C_n(\omega).$$