Lecture 4 Gaussian Processes

Def A SP $(X_t)_{t\in\mathbb{T}}$ is a Gaussian process if $\forall\, t_1 \dots t_n \in \mathbb{T}$, $\forall n$

$(X_{t_1}, \dots X_{t_n})$ is a Gaussian vector

(centered if $\mathbb{E} X_t = 0 \ \forall t$) □

Def $c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ $c(t,s) = c(s,t) \ \forall s,t \in \mathbb{T}$.

we say c is PSD if $\forall\, t_1 \dots t_n \in \mathbb{T}$

$$C_n = (c(t_i, t_j))_{i,j \le n} \quad \text{is PSD}$$

Rmk 1 If $(X_t)_{t\in\mathbb{T}}$ is a SP $\mathbb{E}(X_t^2) < \infty \ \forall t$

then $c(t,s) = \mathrm{Cov}(X_t; X_s)$ is PSD.

Rmk 2 Given $c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ PSD, $m : \mathbb{T} \to \mathbb{R}$

there exists a GP (X_t) with covariance c and mean m $(\mathbb{E} X_t = m(t))$.

The law of (X_t) (on $\mathbb{R}^{\mathbb{T}}$) is unique.

Proof $\forall\, t_1 \dots t_n$ $\quad \underline{m}_n = (m(t_1) \dots m(t_n))$

$C_n = (c(t_i, t_j))_{i,j \le n}$

$$\mu_{t_1 \dots t_n} := N(\underline{m}_n, C_n).$$

Easy to check (μ) are consistent □

Rmk 3 If $\mathcal{H}$ is a Hilbert space.

$\forall\, t_1 t_2 \in \mathcal{H}$ define $c(t_1,t_2) = \langle t_1, t_2 \rangle_{\mathcal{H}}$

$$\sum_{i,j=1}^{n} c(t_i,t_j)\, v_i v_j = \Big\| \sum_{i=1}^{n} v_i t_i \Big\|_{\mathcal{H}}^2 \geq 0$$

so c is PSD.

$\Rightarrow \exists$ GP indexed by $\mathbb{T} = \mathcal{H}$.

with covariance c.

$\mathcal{H} = (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ consider $m=0$

$t \in \mathbb{R}^d$ $\quad \mathbb{E}(X_{t_1} X_{t_2}) = \langle t_1, t_2 \rangle$

$t = e_i$ $\quad \mathbb{E}(X_{e_i} X_{e_j}) = 1_{i=j}$

$$X_t = \langle t, G \rangle \qquad G \sim N(0, I_d)$$

[$\mathcal{H} = L^2([0,1], \mathrm{Leb})$

White noise process.

Lemma Assume $m=0$. $\mathbb{T} = [a,b]^d$. If

(*) $\quad c(t,t) + c(s,s) - 2c(t,s) \leq C \|t-s\|^{2\bar{\gamma}}$

Then $\exists\, (X_t)$ GP with cov c which is γ-Hölder $\forall\, \gamma < \bar{\gamma}$.

Rmk (*) means

$$\mathbb{E}[|X_t - X_s|^2] \leq C \|t-s\|^{2\bar{\gamma}}$$

□

Proof Apply KC

wts
$\hookrightarrow$ $\mathbb{E}[|X_t - X_s|^\alpha] \leqslant C \, \|t-s\|^{\beta+d}$

$\gamma < \frac{\beta}{\alpha} = \frac{\beta+d}{\alpha} - \frac{d}{\alpha}$

$\alpha = 2k$

$$\mathbb{E}[|X_t - X_s|^{2k}] \leqslant C_k \, \mathbb{E}[|X_t - X_s|^2]^k$$

$$\leqslant C_k \cdot [c(t,t) + c(s,s) - 2c(t,s)]^k$$

$$\leqslant C_k' \, \|t-s\|^{2\bar{\gamma} \cdot k}$$

$$\gamma < \frac{2\bar{\gamma}k}{2k} - \frac{d}{2k} = \bar{\gamma} - \frac{d}{2k}$$ take k large enough

$\square$

Lemma $(X^{(\ell)})_{t \in \mathbb{T}}$, $(X_t)_{t \in \mathbb{T}}$ with $(X_t^{(\ell)})$ GP

If $\forall t \in \mathbb{T}$

$$\lim_{n \to \infty} \mathbb{E}[(X_t^{(\ell)} - X_t)^2] = 0$$

then (X_t) is a GP $\quad m(t) = \lim_{\ell \to \infty} m_\ell(t)$

$c(t,s) = \lim_{\ell \to \infty} c_\ell(t,s)$ $\square$

Proof Fix n, $t_1 \dots t_n$, $\mu_{t_1 \dots t_n}$ fdd of (X_t)

$$\mu^{(\ell)}_{t_1 \dots t_n} = N(\underline{m}_n^{(\ell)}, \underline{C}_n^{(\ell)})$$

$\llcorner (c^{(\ell)}(t_i, t_j))_{i,j \leqslant n}$

$$\underline{X}_n^{(\ell)} = \pi_{t_1 \dots t_n}(X^{(\ell)}) \sim \mu^{(\ell)}_{t_1 \dots t_n}$$

$$\underline{X}_n = \pi_{t_1 \dots t_n}(X)$$

By assumption

$$\mathbb{E}\left[\|\underline{X}_n^{(\ell)}-\underline{X}_n\|_2^2\right]=\sum_{i=1}^{n}\mathbb{E}\left[(X_{t_i}^{(\ell)}-X_{t_i})^2\right]\xrightarrow[\ell\to\infty]{}0.$$

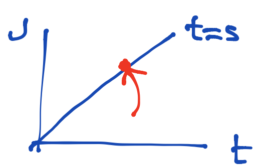
$\Rightarrow$ sequence of G vectors converging in L^2. The limit is Gaussian with covariance = limit of covariances. $\square$

$\gamma_k \uparrow \bar{\gamma}$ $\qquad \mathcal{N}_k \qquad \mathcal{N}=\bigcup_k \mathcal{N}_k$

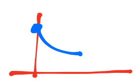
For each k construct X_k γ_k Hölder and $X_k(\omega)=\bar{X}(\omega)$ $\forall\ \omega\notin\mathcal{N}_k$

Set $\quad X=\bar{X} \quad$ for $\omega\notin\mathcal{N}$

$\qquad X=0 \quad$ ow. $\square$



$c(t,t)+c(s,s)-2c(t,s) \leq c_0\|t-s\|^{2\bar{\gamma}}$



<u>Rmk</u> A GP $(X_t)_{t\geq 0}$ has indep increments iff $\forall\, t_1\leq t_2<t_3$

$$\mathrm{Cov}(X_{t_1}, X_{t_3}-X_{t_2})=0 \qquad \square$$

[General def was

$X_{t_3}-X_{t_2}$ indep of $\mathcal{F}^X_{t_2}=\sigma(\{X_t : t\leq t_2\})$

$=\sigma(\{X_{s_1}\in A_1 \ldots X_{s_n}\in A_n : n, s_i, A_i\})$

sufficient to check $\forall n,\ s_1 \ldots s_n \leq t_2$

$X_{t_3} - X_{t_2}$ indep of $(X_{s_1} \ldots X_{s_n})$

For GP uncorrelated $\Rightarrow$ indep.

suff to check

$X_{t_3} - X_{t_1}$ uncorrel with X_{s_i} $\quad i \leq n$ $\square$]

$\mathrm{Cov}(X_{t_1}, X_{t_3} - X_{t_2}) = 0 \quad \forall t_1 \leq t_2 < t_3$

$(*)$ $\quad c(t_1, t_3) = c(t_1, t_2) \qquad$ set $t_1 = t_2 = t$, $t_3 = t'$

$$c(t, t') = \underbrace{c(t,t)}_{g(t)} \qquad \forall t' > t$$

$$\boxed{c(t,t') = g(t \wedge t')} \qquad \forall t\, t' \in \mathbb{R}_{\geq 0}$$

viceversa, this implies indep increm.

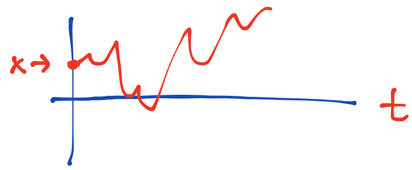
<u>Definition</u> Wiener process / Brownian motion is GP on $\mathbb{T} = \mathbb{R}_{\geq 0}$ with covariance $c(t,t') = t \wedge t'$, $m(t) = x \in \mathbb{R} \quad \forall t$ with cont. sample paths $\square$

<u>Rmk</u> - Has indep increments

- $c(t,t')$ 1-Lipschitz as $t \to t'$
  2γ-Hölder $\gamma = 1/2$

$\Rightarrow$ BM is γ-Hölder $\forall \gamma < 1/2$.

$\Rightarrow$ Can be shown not γ-Hölder $\forall \gamma \geq 1/2$.



Def θ_s := Shift by $s \in \mathbb{R}_{\geq 0}$

$$\theta_s : \mathbb{R}^{[0,\infty)} \to \mathbb{R}^{[0,\infty)}$$
$$x \longmapsto \theta_s(x) \qquad \theta_s(x)(t) = x(t+s) \quad \square$$

It is measurable on $\mathcal{B}^{[0,\infty)}$

(X_t) with law P_X, $P_X \circ \theta_s^{-1}$ law of shifted process

$$P_X \circ \theta_s^{-1}(B) = \mathbb{P}(\theta_s(X) \in B)$$

Def $(X_t)_{t \in [0,\infty)}$ is stationary if $P_X = P_X \circ \theta_s^{-1}$.

Rmk Stationary $\boxed{\text{iff.}}$ $\forall n, t_1 .. t_n$

$$(X_{t_1+s}, \dots X_{t_n+s}) \overset{d}{=} (X_{t_1}, \dots X_{t_n})$$

$$P_{X_{t_1+s} .. X_{t_n+s}} = P_{X_{t_1} .. X_{t_n}} \quad \square$$

In particular stationarity implies

$c(t,s) = \mathrm{Cov}(X_t, X_s) \qquad m(t) = \mathbb{E} X_t$

$$\left[\begin{array}{ll} c(t_1+s, t_2+s) = c(t_1,t_2) = r(|t_1-t_2|) & (*) \\ m(t+s) = m(t) = m_0 & (**) \end{array}\right.$$

Def: $(X_t)_{t\in[0,\infty)}$ $\mathbb{E}X_t^2 < \infty$ second order station.
if (*), (**). (L^2-station, weakly st.)

In general L^2-stationarity $\not\Rightarrow$ stationarity
For GP two notions are equiv.

Ex: BM not stationary!

$t>s$ $\mathbb{E}[(X_t - X_s)^2] = t - 2(t\wedge s) + s = t-s$

$$\mathbb{E}(X_t^2) = \mathbb{E}(X_0^2) + \mathbb{E}[(X_t - X_0)^2] = x^2 + t$$

Def $(X_t)_{t\in[0,\infty)}$ stationary increm.
if $\forall\, t, t', h \geq 0$

$$X_{t+h} - X_t \overset{d}{=} X_{t'+h} - X_{t'}$$

Ex For BM $X_{t+h} - X_t \sim N(0,h)$ □

Indep incr $c(t,t') = g(t\wedge t')$ ✓

(W_t) a BM started at $x=0$
h a non decr funct: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$X_t := W_{h(t)}$. centered Gaussian process

$$\mathbb{E}(X_{t_1} X_{t_2}) = \mathbb{E} W_{h(t_1)} W_{h(t_2)} = h(t_1) \wedge h(t_2)$$
$$= h(t_1 \wedge t_2).$$

$$\mathbb{E}(X_t^2) = \mathbb{E}(X_t)^2 + Var(X_t) = m(t)^2 + c(t,t)$$

$$= x^2 + t$$

$$\mathbb{E}[|X|^{2k}] \leq C_k \, \mathbb{E}(X^2)^k \qquad (*)$$

$X \sim N(0, \tau^2)$ $\qquad X = \tau \cdot G \qquad G \sim N(0,1)$

$$\mathbb{E}(|X|^{2k}) = \mathbb{E}(\tau^{2k}|G|^{2k}) = \underbrace{\mathbb{E}(|G|^{2k})}_{C_k} \cdot \mathbb{E}(X^2)^k$$

τ^2 subgaussian. $\qquad \mathbb{E}\, e^{\lambda X} \leq e^{\lambda^2 \tau^2 / 2} \quad \forall \lambda \in \mathbb{R}$

$$\mathbb{P}(|X| \geq t) \leq 2 e^{-t^2 / c\tau^2} \quad (e^{-t/\tau})$$

If X is τ^2 subg. $\tau^2 \leq c \cdot \mathbb{E}(X^2)$ $\qquad (*)$

then $(*)$ holds

$$\mathbb{E}\,|X|^{2k} = \int_0^\infty t^{2k-1} \mathbb{P}(|X| \geq t)\, dt \leq C_k \tau^{2k}$$

$$c(t,t) + c(s,s) - 2c(t,s) \leq c_0 \, \|t-s\|^{2\gamma}.$$

For ant. $\underbrace{\mathbb{E}(|X_t - X_s|^2)}_{t-s} \leq c_0 \, \|t-s\|^{2\gamma}$