Lecture 5 Filtrations, stopping times.

$\mathbb{T} = [0,\infty)$, $[0,\infty]$ unless stated $(\Omega, \mathcal{F}, \mathbb{P})$

Def A filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ is a seq of σ-algebras

$\mathcal{F}_t \subseteq \mathcal{F}$ st $s \le t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$

$\mathcal{F}_\infty = \sigma\left(\bigcup_{t=0}^{\infty} \mathcal{F}_t\right)$ $\mathcal{F}_t \uparrow \mathcal{F}_\infty$

A SP $(X_t)_{t\ge 0}$ is adapted to $(\mathcal{F}_t)$ if

$X_t \in m\mathcal{F}_t \quad \forall t \ge 0.$

Canonical filtration

$$\mathcal{F}_t^X = \sigma(\{X_s : s \in [0,t]\})$$

$$\mathcal{F}_\infty^X = \sigma(\{X_t : t \in [0,\infty)\})$$

$$\boxed{\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s} \qquad \mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$$

both of these are filtrations

$\mathcal{G}_t := \mathcal{F}_{t+}$

$(\mathcal{F}_t)$ is right continuous if $\forall t$

$\mathcal{F}_{t+} = \mathcal{F}_t .$

We always assume $(\mathcal{F}_t)$ complete $\overset{\text{nullsets.}}{\mathcal{N}} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_t$

I want to be able to talk about $X_{T(\omega)}(\omega)$

Def $(X_t)_{t\in[0,\infty)}$ is progressively measurable if adap.
$\forall\, t$

$$X : [0,t]\times\Omega \to \mathbb{R}$$
$$(s,\omega) \mapsto X_s(\omega)$$

is measurable on $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$ $\square$

Lemma If (X_t) is right continuous, adapted then it is prog. meas.

Proof Same as for measurab.

Def $\tau : \Omega \to [0,\infty]$ is an $(\mathcal{F}_t)$ stopping time if $\forall t$ $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_t \ \forall t\}.$$

Rmk $\{\tau < t\} \in \mathcal{F}_t$

Why? Take $t_k \uparrow t$. Then

$$\{\tau < t\} = \bigcup_{k=1}^{\infty} \underbrace{\{\tau \leq t_k\}}_{\in \mathcal{F}_{t_k} \subseteq \mathcal{F}_t} \implies \{\tau < t\} \in \mathcal{F}_t$$

Rmk $\{\tau = \infty\} = \Big(\bigcup_{k=1}^{\infty} \underbrace{\{\tau \leq k\}}_{\in \mathcal{F}_k}\Big)^c \in \mathcal{F}_\infty$

$\mathcal{F}_{t+} \supseteq \mathcal{F}_t$

<u>Def</u> A stopping time for $\mathcal{F}_{t+}$ is called Markov time.

In general $\mathcal{F}_{t+} \neq \mathcal{F}_t$

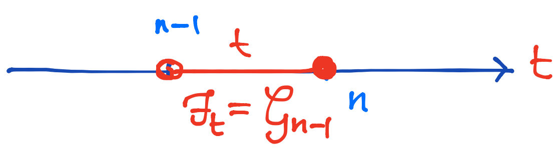
(Stopping $\Rightarrow$ Markov time not vice versa in general)

<u>Example</u> $(\mathcal{G}_n)_{n \in \mathbb{Z}_{\geq 0}}$ discrete time filtration

$\Omega = \mathbb{R}^{\mathbb{Z}_{\geq 0}}$ $\quad \mathcal{G}_n := \sigma(\{Z_0, Z_1, \dots Z_n\})$ $\quad Z_i = \omega_i$

$\mathcal{F} = \underline{\mathcal{B}}^{\mathbb{Z}_{\geq 0}}$ $\qquad \mathcal{B} = \mathcal{B}_{\mathbb{R}}$

$\mathcal{F}_t := \mathcal{G}_{\lceil t \rceil - 1}$



$\mathcal{F}_n = \mathcal{G}_{n-1}$

$\mathcal{F}_{n+\epsilon} = \mathcal{G}_n$

$$\mathcal{F}_{n+} = \bigcap_{\epsilon > 0} \mathcal{F}_{n+\epsilon} = \underline{\mathcal{G}_n} \neq \underline{\mathcal{F}_n}.$$

$$\tau(\omega) = \inf\{ n : \omega_n \geq a \}$$

$\{\tau(\omega) \leq n\} \in \mathcal{G}_n \quad ; \quad \notin \mathcal{G}_{n-1}$

$\{\tau(\omega) \leq n\} \in \mathcal{F}_{n+} \quad ; \quad \notin \mathcal{F}_n.$

Markov time but not stopping

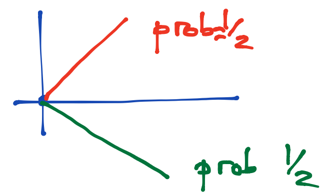
$\mathcal{F}_t \quad ; \quad \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s \qquad$ in general $\mathcal{F}_t \neq \mathcal{F}_{t+}$

$\mathcal{G}_t = \mathcal{F}_{t+} \quad ; \quad \mathcal{G}_{t+} = \bigcap_{s>t} \mathcal{G}_s \qquad \mathcal{G}_{t+} = \mathcal{G}_t .$

Continuous sample paths $\not\Rightarrow$ Right cont $\mathcal{F}_t^X$

<u>Example</u> $\quad (\Omega, \mathcal{F}, \mathbb{P}) = (\{+1,-1\}, 2^{\{+1,-1\}}, \text{Unif})$

$$X_t(\omega) = \omega t$$



$\mathcal{F}_0^X = \sigma(\{X_0\}) = \{\emptyset, \Omega\}$

$\mathcal{F}_t^X \supseteq \sigma(\{X_t\}) = 2^\Omega \qquad \mathcal{F}_t = 2^\Omega$

$\mathcal{F}_{0+}^X = 2^\Omega \neq \mathcal{F}_0^X .$

<u>Proposition</u> (1) τ is Markov for $\mathcal{F}_t$ iff

$\{\tau < t\} \in \mathcal{F}_t \ \forall t \quad$ iff $\quad \tau \wedge t$ is $\mathcal{F}_t$ meas $\forall t$

(2) τ a Markov time

$$\mathcal{F}_{\tau+} := \{A \in \mathcal{F}_\infty : \{\tau \leq t\} \cap A \in \mathcal{F}_{t+} \ \forall t\}$$
$$= \{A \in \mathcal{F}_\infty : \{\tau < t\} \cap A \in \mathcal{F}_t \ \forall t\}$$

(3) τ stopping $\Rightarrow \quad \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau+}$, if right cont then $\mathcal{F}_\tau = \mathcal{F}_{\tau+}$.

(4) If τ stopping, then τ is $\mathcal{F}_\tau$ measur.

(5) $\tau = t$ constant is a stopping time

$$\mathcal{F}_\tau = \mathcal{F}_t, \quad \mathcal{F}_{\tau+} = \mathcal{F}_{t+}.$$

(6) τ, σ stopping $\Rightarrow$ $\tau \vee \sigma, \tau \wedge \sigma$ stopping

(7) $(\tau_n)_{n\in\mathbb{N}}$ stopping $\Rightarrow$ $\sup_n \tau_n$ stopping

(8) $(\tau_n)_{n\in\mathbb{N}}$ Markov $\Rightarrow$ $\inf_n \tau_n$ Markov.

Proof of (1) τ Markov iff $\{\tau<t\}\in\mathcal{F}_t \ \forall t$

($\Rightarrow$) $\{\tau<t\} = \bigcup_{k=1}^{\infty} \underbrace{\{\tau\leq t_k\}}_{\in \mathcal{F}_{t_k+}\subseteq \mathcal{F}_t} \in \mathcal{F}_t \qquad t_k \uparrow t, \quad t_k<t$

($\Leftarrow$) $\{\tau\leq t\} = \bigcap_{k=1}^{\infty} \{\tau < s_k\} \qquad s_k \downarrow t$

Fix $\varepsilon>0$

$\exists k_0$: $s_k<t+\varepsilon \ \forall k\geq k_0$

$$= \bigcap_{k=k_0}^{\infty} \underbrace{\{\tau<s_k\}}_{\in \mathcal{F}_{s_k}\subseteq \mathcal{F}_{t+\varepsilon}} \in \mathcal{F}_{t+\varepsilon}$$

$$\{\tau\leq t\} \in \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_{t+}$$

τ Markov time $\Leftrightarrow$ $\tau\wedge t$ is $\mathcal{F}_t$ meas $\forall t$

($\Leftarrow$) $\{\tau<t\} = \{\underbrace{\tau\wedge t}<t\} \in \mathcal{F}_t$

($\Rightarrow$) Need to check $\{\tau\wedge t<s\}\in\mathcal{F}_t \ \forall t,s$

$t<s$ obvious ; $t\geq s$ $\{\tau\wedge t<s\} = \{\tau<s\}\in\mathcal{F}_s\subseteq\mathcal{F}_t$

(8) $(\tau_n)_{n\in\mathbb{N}}$ Markov times $\Rightarrow$ $\inf_{n\in\mathbb{N}} \tau_n$ Markov

$\{\inf_n \tau_n < t\} \in \mathcal{F}_t$ $\leftarrow$ Need to prove this

$$\{\inf_n \tau_n < t\} = \bigcup_n \underbrace{\{\tau_n < t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t$$

Does not hold if $\leq$ $\square$

$$\{\inf_n \tau_n \leq t\} \neq \bigcup_n \{\tau_n \leq t\}$$

$\tau_n = 1/n \quad t = 0$ $\square$

<u>Proposition</u> $(X_t)_{t\geq 0}$ progr. meas. τ an $\mathcal{F}_t$ stopping then $Y_t = X_{t\wedge\tau}$ is progr meas.

In particular if either $\tau < \infty$ a.s or $X_\infty \in m\mathcal{F}_\infty$ then $X_\tau \in m\mathcal{F}_\tau$. $\blacksquare$

<u>Proof</u> $\mathcal{X} = [0,t] \times \Omega$,

$\mathcal{G} = \overline{\mathcal{B}}_{[0,t]} \times \mathcal{F}_t$

(Here we prove in part. $X_{\tau\wedge t} = Y_t \in m\mathcal{F}_t$)

$$X(s,\omega) = X_s(\omega) \;;\; X : (\mathcal{X}, \mathcal{G}) \to (\mathbb{R}, \mathcal{B})$$

$$Y(s,\omega) := Y_s(\omega)$$

$$Y(s,\omega) = X(s\wedge\tau(\omega), \omega) = \underbrace{X \circ h(s,\omega)}$$ ✓

$$h(s,\omega) = (\underbrace{s\wedge\tau(\omega)}_{\in m\mathcal{F}_s \subseteq m\mathcal{F}_t}, \omega) \in m\mathcal{G} \quad \Rightarrow Y \in m\mathcal{G}$$

If either $\tau<\infty$ a.s. or $X_\infty \in m\mathcal{F}_\infty$, then $X_\tau \in m\mathcal{F}_\tau$

Fix $B \in \mathcal{B}_{\mathbb{R}}$

Need to check $X_\tau^{-1}(B) \cap \{\tau \le t\} \in \mathcal{F}_t$ $(*)$

$(*)$ $X_\tau^{-1}(B) \cap \{\tau\le t\} = \{\omega: \ X_{\tau(\omega)}(\omega) \in B;\ \tau(\omega)\le t\}$

$$= \{\omega: \ X_{\tau(\omega)\wedge t} \in B,\ \tau(\omega)\le t\}$$

$$= \underbrace{\{\omega: X_{\tau(\omega)\wedge t} \in B\}}_{\in \mathcal{F}_t \text{ by first part}} \cap \underbrace{\{\omega: \tau(\omega)\le t\}}_{\substack{\in \mathcal{F}_t \\ \text{since } \tau \\ \text{stopping}}}$$

For $t=\infty$

$= \{\omega: X_{\tau(\omega)} \in B\}$

Need to check $(*)$ for $t=\infty$

$X_\tau^{-1}(B) \in \mathcal{F}_\infty$ $\quad \underbrace{[X_\tau^{-1}(B) = [X_\tau^{-1}(B) \cap \{\tau<\infty\}]}$

$\cup [X_\tau^{-1}(B) \cap \{\tau=\infty\}]$

$$\| $$

$$\bigcup_{n=1}^{\infty} \underbrace{[X_\tau^{-1}(B) \cap \tau^{-1}([0,n])]}_{\mathcal{F}_n \subseteq \mathcal{F}_\infty} \cup [\underbrace{X_\infty^{-1}(B)}_{\in \mathcal{F}_\infty} \cap \underbrace{\tau^{-1}(\infty)}_{\mathcal{F}_\infty}]$$

$$Y_t = X_{\tau\wedge t} \qquad Y_\infty = \begin{cases} 0 & \text{if } \tau=\infty \\ \lim_{t\to\infty} Y_t & \text{ow} \end{cases}$$

$$X_\tau = \underbrace{Y_\infty}_{\in m\mathcal{F}_\infty} 1_{\tau<\infty} + \underbrace{X_\infty}_{\in m\mathcal{F}_\infty} \underbrace{1_{\tau=\infty}}_{\in m\mathcal{F}_\infty} \quad \in m\mathcal{F}_\infty.$$