Lecture 6

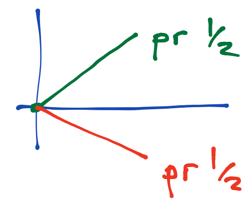
Hitting times are not always stopping times ($\mathcal{F}_t^X$)

$B \in \mathcal{B}$ $(X_t)_{t\geq 0}$ SP

$$\tau_B := \inf\{t \geq 0 : X_t \in B\}$$

Example $(\Omega, \mathcal{F}, \mathbb{P}) = (\{+1,-1\}, 2^{\{+1,-1\}}, \text{Unif})$

$X_t(\omega) = \omega t \qquad t \in [0,\infty)$



$B = (0,\infty) \qquad \tau_B$

$$\tau_B(\omega) = \begin{cases} 0 & \text{if } \omega = +1 \\ \infty & \text{if } \omega = -1 \end{cases}$$

$\{\tau_B(\omega) \leq 0\} \notin \mathcal{F}_0^X = \{\emptyset, \Omega\}, \in \mathcal{F}_t^X \quad t > 0$

Prop. $(X_t)_{t\geq 0}$ $\mathcal{F}_t$ adapted.

(1) B open (X_t) right cont $\Rightarrow \tau_B$ is $\mathcal{F}_t$ Markov

(2) B close (X_t) cont $\Rightarrow \tau_B$ is $\mathcal{F}_t$ stopping $\square$

Proof (1)

$$\{\tau_B < t\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{\omega : X_s(\omega) \in B\}$$

Indeed

$\tau_B(\omega) < t \Rightarrow \exists\ s_* < t$ st $X_{s_*}(\omega) \in B$

can take $s_k \downarrow s_* \quad s_k \in \mathbb{Q}\ \forall k$

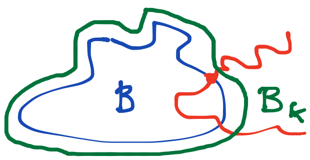
By r.c. of X_t $\quad X_{s_k}(\omega)\in B \quad \forall k\geq k_0$
since B open

$$\Rightarrow \quad \{\tau_B<t\} \underset{=}{\subseteq} \bigcup_{s\in[0,t)\cap\mathbb{Q}} \{\omega : \underbrace{X_s(\omega)\in B}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t}\}$$

$$\Rightarrow \quad \{\tau_B<t\} \in \mathcal{F}_t \quad \Rightarrow \tau_B \text{ Markov.}$$

(2) X_t cont B closed $\Rightarrow \tau_B$ stopping

$$A := \{\tau_B \leq t\} = \bigcap_{k=1}^{\infty} \{\tau_{B_k} < t\} =: A_k \quad \circledast$$ by continuity of X



$$B_k := \{x : d(x,B) < 1/k\}$$

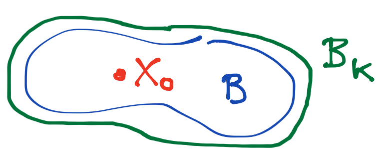
B_k open $\quad B = \bigcap_k B_k.$

τ_{B_k} is Markov (by point (1))

$$\Rightarrow \{\tau_{B_k}<t\} \in \mathcal{F}_t \quad \Rightarrow \quad \{\tau_B \leq t\} \in \mathcal{F}_t$$

$X_0(\omega) \in B \Rightarrow \quad \omega\in A, \quad \omega\in A_k \ \forall k$

$X_0(\omega)\in B_k$



$\tau_B(\omega)=0$

$\tau_{B_k}(\omega)=0$

$$\{\tau_B \leq t\} = \{X_0\in B\} \cup \left[\bigcap_{k=1}^{\infty} \{\tau_{B_k}<t\}\right]$$

Def (X_t) $\mathcal{F}_t$ adapted $X_t \in L^1$ $\forall t$

1. X_t is a MG if $t > s \Rightarrow \mathbb{E}[X_t | \mathcal{F}_s] = X_s$.
2. X_t is a subMG if $t < s \Rightarrow \mathbb{E}[X_t | \mathcal{F}_s] \geq X_s$
3. supM $\leq$

Example (X_t) indep. increments $[X_t - X_s$ indep of $\mathcal{F}_s^X]$ $t > s$

* $Y_t = X_t - \mathbb{E}X_t$ is a MG

$$\mathbb{E}[Y_t - Y_s | \mathcal{F}_s] = \mathbb{E}[X_t - X_s | \mathcal{F}_s] - \mathbb{E}[X_t - X_s] = 0$$

* If $X_t \in L^2$ $Y_t = X_t^2 - \mathbb{E}X_t^2$ is a MG
* If $\mathbb{E}e^{\theta X_t} < \infty$ for some $\theta \in \mathbb{R}$, then

$$Z_t = \frac{e^{\theta X_t}}{\mathbb{E}e^{\theta X_t}}$$ is a MG.

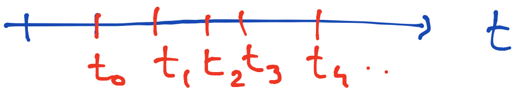
- $(Y_n, \mathcal{G}_n)_{n \geq 1}$ a discr. time MG

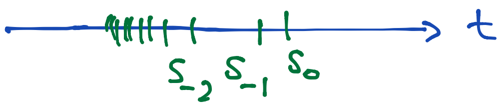
  $X_t := Y_{\lfloor t \rfloor}$ $\mathcal{F}_t = \mathcal{G}_{\lfloor t \rfloor}$

  $\Rightarrow (X_t, \mathcal{F}_t)$ is a MG. (sub/sup)

- $(X_t, \mathcal{F}_t)$ MG $(t_0 \leq t_1 \leq t_2 \ldots)$

  $Y_n := X_{t_n}$ $\mathcal{G}_n := \mathcal{F}_{t_n} \Rightarrow (Y_n, \mathcal{G}_n)$ is MG





$s_0 \geq s_{-1} \geq \dots$

$$(Y_{-k}, \mathcal{G}_{-k})_{k\in\mathbb{N}} = (X_{s_{-k}}, \mathcal{F}_{s_{-k}})_{k\in\mathbb{N}}$$

is a reverse MG (sub/sup)

$$\mathbb{E}(Y_{-k} \mid \mathcal{G}_{-k-1}) = Y_{-k-1}.$$

$\square$

<u>Thm</u> (Doob's max ineq) $(X_t)_{t\geq 0}$ $\boxed{\text{right cont}}$ sub MG.

$$x>0 \Rightarrow \mathbb{P}\big(\underline{\sup_{0\leq s\leq t} X_t} \geq x\big) \leq \frac{1}{x}\mathbb{E}\big((X_t)_+\big)$$

$\square$

<u>Proof</u> $M = \sup_{0\leq s\leq t} X_s$. Suff. to show $\forall\, y>0$

$$\boxed{y\,\mathbb{P}(M>y) \leq \mathbb{E}(X_t 1_{\{M>y\}})}$$

Indeed $\mathbb{E}(X_t 1_{M>y}) \leq \mathbb{E}((X_t)_+)$

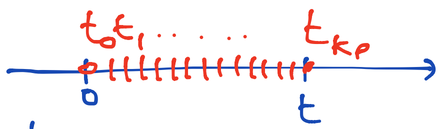
and taking $y\uparrow x$ $\quad y\,\mathbb{P}(M>y) \to x\,\mathbb{P}(M\geq x)$

Want to show $y\,\mathbb{P}(M>y) \leq \mathbb{E}[X_t \mathbb{1}_{M>y}]$

$$\mathbb{D}_\ell = \{0 = t_0 < t_1 < \dots < t_{k_\ell} = t\}$$

$$0 < t_{j+1} - t_j \leq 2^{-\ell} \qquad \forall j \leq k_\ell$$

$\mathbb{D}_\ell \subseteq \mathbb{D}_{\ell+1}$



$t_j = j/2^\ell$

$\mathbb{D}_\infty = \bigcup_{\ell=1}^{\infty} \mathbb{D}_\ell$ dense in $[0,t]$

$$M(\ell) := \max_{t \in \mathbb{D}_\ell} X_t$$

(X_{t_j}) is a discr. time sub MG

$$x\,\mathbb{P}(M(\ell) \geq x) \leq \mathbb{E}(X_t \mathbb{1}_{M(\ell) \geq x})$$

$x \downarrow y$ dominated

$$y\,\mathbb{P}(M(\ell) > y) \leq \mathbb{E}(X_t \mathbb{1}_{M(\ell) > y}) \quad (*)$$

As $\ell \to \infty$ $M(\ell) \uparrow M$ right continuity.

(in general $M(\ell) \uparrow M(\infty) = \sup_{t \in \mathbb{D}_\infty} X_t$)

$$\mathbb{1}_{M(\ell) > y} \uparrow \mathbb{1}_{M>y}$$

(This works bec. we replaced $\geq$ with $>$)

By dom. $(*)$ implies

$$y\,\mathbb{P}(M>y) \leq \mathbb{E}(X_t \mathbb{1}_{M>y})$$

$\square$

Thm (Doob convergence) $(X_t)_{t\geq 0}$ right cont. sub MG such that $\boxed{\sup_t \mathbb{E}[(X_t)_+] < \infty}$.

Then $X_t \xrightarrow{a.s.} X_\infty$ $\mathbb{E}|X_\infty| \leq \liminf_{t\to} \mathbb{E}|X_t| < \infty$

[The following are equiv for sub MG X_t

(i) $\lim_{t\to\infty} \mathbb{E}|X_t|$ exists finite

(ii) $\sup_t \mathbb{E}|X_t| < \infty$

(iii) $\liminf_t \mathbb{E}|X_t| < \infty$

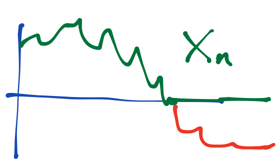
(iv) $\lim_{t\to\infty} \mathbb{E}(X_t)_+$ exists finite

(v) $\sup_t \mathbb{E}(X_t)_+ < \infty$]

This thm does not imply conv in L^1 !!!

__Example__ S_n = symm RW started at $x \in \mathbb{Z}_{>0}$

$X_n = S_{n\wedge T_0}$ $T_0 := \inf\{n : S_n = 0\}$.



X_n is a MG.

$\mathbb{E}X_n = \mathbb{E}X_0 = x$

$\overset{x}{\overset{\|}{\mathbb{E}X_n}} \not\to 0$ $X_n \xrightarrow{a.s} X_\infty$ $X_\infty = 0$ a.s.

Def $(X_t, \mathcal{F}_t)_{t\geq 0}$ a sub MG, $\mathcal{F}_\infty = \sigma(\bigcup_t \mathcal{F}_t)$
is <u>right closable</u> or <u>has a last elem</u>
$(X_\infty, \mathcal{F}_\infty)$ if $X_\infty \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$
and $\forall t$ $X_t \leq \mathbb{E}(X_\infty | \mathcal{F}_t)$ a.s.

[for MG $X_t = \mathbb{E}(X_\infty | \mathcal{F}_t)$
i.e. X_t is a Doob MG]

Prop $(X_t)_{t\geq 0}$ <u>right cont</u> <u>non-neg</u> sub MG.
The following are equiv:

(1) $(X_t)_{t\geq 0}$ is UI

(2) $X_t \xrightarrow{L^1} X_\infty$ ✓

(3) $X_t \xrightarrow{as} X_\infty$ with X_∞ a last elem.

[For (1) ⇒ (2) ⇒ (3) non negat. not needed]

Proof

$\boxed{(1) \Rightarrow (2)}$ (X_t) UI $\Rightarrow$ bounded in L^1

$\Rightarrow \sup_t \mathbb{E}((X_t)_+) < \infty$. Can apply Doob

$\Rightarrow$ $X_t \xrightarrow{a.s.} X_\infty \overset{UI}{\Longrightarrow} X_t \xrightarrow{L^1} X_\infty$

$\boxed{(2) \Rightarrow (3)}$ I know $X_t \xrightarrow{L^1} X_\infty$, by Doob $X_t \xrightarrow{as} X_\infty$

Want to show X_∞ is a last elem

namely $\mathbb{E}(X_\infty | \mathcal{F}_s) \geq X_s$ a.s.

We know (sub MG) $\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s$ a.s.

$\forall A \in \mathcal{F}_s$ $\mathbb{E}[X_t \mathbb{I}_A] \geq \mathbb{E}[X_s \mathbb{I}_A]$

$$\lim_{t\to\infty} \mathbb{E}[X_t \mathbb{I}_A] = \mathbb{E}[X_\infty \mathbb{I}_A] \quad (\text{bec. } X_t \xrightarrow{L^1} X_\infty)$$

$$\Rightarrow \mathbb{E}[X_\infty \mathbb{I}_A] \geq \mathbb{E}[X_s \mathbb{I}_A] \quad \forall A \in \mathcal{F}_s$$

$$\mathbb{E}[\,[\underbrace{\mathbb{E}[X_\infty | \mathcal{F}_s] - X_s}_{Z \,\in m \mathcal{F}_s}]\,\mathbb{I}_A] \geq 0$$

$\boxed{(3) \Rightarrow (1)}$ We know $X_t \xrightarrow{a.s.} X_\infty$ a last elem

wts X_t is UI

$$\mathbb{E}(X_t) \leq \mathbb{E}[\mathbb{E}(X_\infty | \mathcal{F}_t)] \leq \mathbb{E}(X_\infty)$$

$$\mathbb{P}(X_t \geq M) \leq \frac{1}{M} \mathbb{E} X_t \leq \frac{1}{M} \mathbb{E} X_\infty .$$

$$\lim_{M\to\infty} \sup_t \mathbb{P}(X_t \geq M) = 0$$

$$\mathbb{E}(X_t 1_{X_t \geq M}) \leq \mathbb{E}[\mathbb{E}(X_\infty | \mathcal{F}_t) 1_{X_t \geq M}]$$

$$= \mathbb{E}(X_\infty \underbrace{1_{X_t \geq M}}_{I_A}) \leq \sup_{A : \mathbb{P}(A) = \mathbb{P}(X_t \geq M)} \mathbb{E}(X_\infty I_A)$$

$$\leq \mathbb{E}(X_\infty I_{X_\infty \geq L(M)})$$

$$\mathbb{P}(X_\infty \geq L(M)) \geq \underbrace{\sup_t \mathbb{P}(X_t \geq M)}_{\to 0}$$

Hence $L(M) \to \infty$

$$\sup_t \mathbb{E}(X_t I_{X_t \geq M}) \leq \mathbb{E}(X_\infty I_{X_\infty \geq L(M)}) \overset{M \to \infty}{\downarrow 0}$$

$\square$