Lecture 7 Optional stopping theorem

Thm $(X_n, \mathcal{F}_n)$ a sub MG $n \in \mathbb{Z}_{\geq 0}$ such that

$X_n \leq \mathbb{E}(X_\infty | \mathcal{F}_n)$ $\forall n$. for some $X_\infty \in L^1$

Then for all stopping times θ, τ. such that $\theta \leq \tau$ a.s.

$\mathbb{E}(X_\tau), \mathbb{E}X_\theta$ exist and

$$\mathbb{E}(X_\tau) \geq \mathbb{E}(X_\theta) \geq \mathbb{E}(X_0)$$

$\square$

Continuous time:

Thm $(X_t, \mathcal{F}_t)$ a right. cont sub MG with a last element X_∞ ($X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ $\mathbb{E}(X_\infty | \mathcal{F}_t) \geq X_t$ $\forall t$) . Two Markov times $\tau \geq \theta$ a.s. then $\mathbb{E}X_\tau, \mathbb{E}X_\theta$ exist and

$$\mathbb{E}(X_\tau) \geq \mathbb{E}(X_\theta) \geq \mathbb{E}(X_0)$$

$\square$

Note: Most common appl. is MG $\mathbb{E}X_\tau = \mathbb{E}X_\theta$.

<u>Proof</u> Discretization argument.

- $X_\tau \in m\mathcal{F}_{\tau+}$, $X_\theta \in m\mathcal{F}_{\theta+}$

Indeed X_t right cont hence progr. meas.

τ is stopping time for $\mathcal{F}_{t+}$.

Further X_∞ exists. $m\mathcal{F}_\infty$

$\Rightarrow X_\tau \in m\mathcal{F}_{\tau+}$.

<u>Discretize</u> $s_k := k/2^\ell$ $k \in \{0,1,2,\dots \infty\}$.

$\mathcal{G}_k^{(\ell)} := \mathcal{F}_{s_k}$ then $(X_{s_k}, \mathcal{G}_k)$ is a sub MG.

We can apply discrete OST.

$\tau_\ell := \inf\{ s_k : \underline{s_k > \tau} \} = 2^{-\ell}(\lfloor 2^\ell \tau \rfloor + 1)$

$\theta_\ell := \inf\{ s_k : \underline{s_k > \theta} \} = 2^{-\ell}(\lfloor 2^\ell \theta \rfloor + 1)$

τ_ℓ, θ_ℓ are $\mathcal{G}_k^{(\ell)}$ stopping times (check!)

$$\mathbb{E} X_{\tau_\ell} \geq \mathbb{E} X_{\theta_\ell} \qquad (*)$$

Further $\tau_\ell \downarrow \tau$ $\theta_\ell \downarrow \theta$ as $\ell \to \infty$.

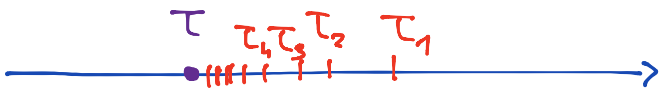
By right continuity

$$X_{\tau_\ell} \to X_\tau, \quad X_{\theta_\ell} \to X_\theta \qquad \text{a.s. as } \ell \to \infty.$$

We know $X_{\tau_\ell} \to X_\tau$

need to show $\mathbb{E}X_{\tau_\ell} \to \mathbb{E}X_\tau$.

Note that



$Z_{-n} := X_{\tau_n}$ for $n \in \mathbb{Z}_{\geq 0}$

$\mathcal{G}_{-n} := \mathcal{F}_{\tau_n}$ is a reverse sub MG.

$\mathbb{E}(Z_{-n} \mid \mathcal{G}_{-n-1}) \geq Z_{-n-1}$ a.s. $\forall n \geq 1$.

($*$ $\mathbb{E}[X_{\tau_n} \mid \mathcal{F}_{\tau_{n+1}}] \geq X_{\tau_{n+1}}$ indeed both τ_n, τ_{n+1} are $\mathcal{G}_k^{(n+1)}$ stopping times and can apply discrete OST $*$)

$Z_{-n} \xrightarrow{a.s.} Z_{-\infty} := X_\tau$.

$\inf\limits_{n \geq 0} \mathbb{E}Z_{-n} = \inf\limits_{n \geq 0} \mathbb{E}X_{\tau_n} \geq \mathbb{E}X_0$.

$\Rightarrow (Z_{-n})$ is UI $\Rightarrow Z_{-n} \xrightarrow{L^1} Z_{-\infty}$.

$X_{\tau_n} \xrightarrow{L^1} X_\tau$. Similarly $X_{\theta_n} \to X_\theta$.

$\mathbb{E}X_{\tau_n} \geq \mathbb{E}X_{\theta_n} \Rightarrow \mathbb{E}X_\tau \geq \mathbb{E}X_\theta$.

$\square$

Corollary (1) If $(X_t)_{t\in[0,\infty]}$ is right cont subMG with last element. and $\tau \geq \theta$ Markov times. Then

$$\mathbb{E}[X_\tau \mid \mathcal{F}_{\theta+}] \geq X_\theta. \qquad \text{a.s.}$$

[= if MG]

If θ is stopping time.

$$\mathbb{E}[X_\tau \mid \mathcal{F}_\theta] \geq X_\theta \qquad \text{a.s.}$$

Proof $A \in \mathcal{F}_{\theta+}$ and define Markov time

$$\eta = \theta I_A + \tau I_{A^c} \qquad \tau \geq \eta \text{ a.s.}$$

$$\mathbb{E}[X_\tau] \geq \mathbb{E}(X_\eta) = \mathbb{E}[X_\theta I_A + X_\tau I_{A^c}]$$

Since $X_\tau, X_\theta \in L^1$

$$\mathbb{E}[(X_\tau - X_\theta) I_A] \geq 0 \qquad \text{since } I_A \in m\mathcal{F}_{\theta+}, \; X_\theta \in$$

$$\mathbb{E}[(\underbrace{\mathbb{E}(X_\tau \mid \mathcal{F}_{\theta+})}_{\in m\mathcal{F}_{\theta+}} - X_\theta) I_A] \geq 0$$

holds for any $A \in m\mathcal{F}_{\theta+} \Rightarrow \mathbb{E}(X_\tau \mid \mathcal{F}_{\theta+}) \geq X_\theta.$

Rmk To check assumption of OST. sufficient either

(1) $X_t \le \mathbb{E}(Y|\mathcal{F}_t)$ for some $Y \in L_1(\Omega, \mathcal{F}, \mathbb{P})$

[difference : not assuming $Y \in m\mathcal{F}_\infty$]

(2) τ bounded a.s. . Eg if $\tau \le T$ for some deterministic T

Can set $\tilde{X}_t = X_{t \wedge T}$.

Apply OST to $\tilde{X}_t$: ok because $\tilde{X}_t$ has last element $\tilde{X}_\infty = X_T$.

Corollary (2) If $(X_t)_{t \ge 0}$ a right cont sub MG and τ is $\mathcal{F}_t$ stopping time . Then

$(X_{t \wedge \tau})_{t \ge 0}$ is a right cont sub MG .

Proof Apply OST. $t_1 < t_2$. $A \in \mathcal{F}_{t_1}$

$$\theta_1 = (\tau \wedge t_1)$$
$$\theta_2 = (\tau \wedge t_1) I_A + (\tau \wedge t_2) I_{A^c}.$$

$$\mathbb{E} X_{\theta_2} \ge \mathbb{E} X_{\theta_1}$$
$$\mathbb{E}[X_{\tau \wedge t_1} I_A + X_{\tau \wedge t_2} I_{A^c}] \ge \mathbb{E}[X_{\tau \wedge t_1}]$$
$$\mathbb{E}[(X_{\tau \wedge t_2} - X_{\tau \wedge t_1}) I_{A^c}] \ge 0$$

$\square$

Given $(\mathcal{X}, \mathcal{G})$ measurable space (state space).
(B. equivalent . $\mathbb{R}^n$. Polish)

A trans. probability is a fct

$p : \mathcal{X} \times \mathcal{G} \to [0,1]$ st.

(1) $\forall x \in \mathcal{X}$, $p(x, \cdot)$ is a prob measure on $(\mathcal{X}, \mathcal{G})$.

(2) $\forall B \in \mathcal{G}$, $p(\cdot, B)$ is a meas. function on $(\mathcal{X}, \mathcal{G})$.

<u>Def</u> A collection of trans' prob $\{p_{t,s} : t \geq s \geq 0\}$. is consistent if $\forall t_1 \leq t_2 \leq t_3$

$$\boxed{p_{t_1 t_3}(x, B) = p_{t_1 t_2} p_{t_2 t_3}(x, B)} \qquad \forall x \in \mathcal{X}, \; B \in \mathcal{G}.$$

and $p_{t,t}(x, B) = I(x \in B)$

$(p_{t,t}(x, \cdot) = \delta_x)$.

$$pq(x, B) = \int_{\mathcal{X}} q(y, B) \, p(x, dy)$$

ν prob measure on $(\mathcal{X}, \mathcal{G})$

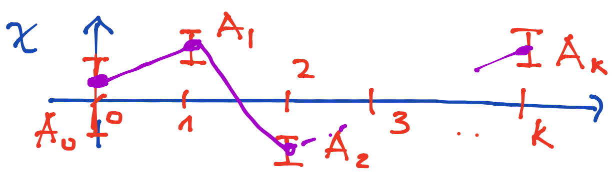
$p_1, p_2, \dots p_K$ trans. prob.

$\nu \otimes p_1 \otimes \dots \otimes p_K$ is the unique prob measure on $(\mathcal{X}^{K+1}, \mathcal{G}^{K+1})$

such that

$$\nu \otimes p_1 \otimes \dots \otimes p_K (A_0 \times A_1 \times \dots \times A_K) =$$

$$= \int_{A_0} \underbrace{p_1(x_0, \cdot)}_{\nu'} \otimes p_2 \otimes \dots \otimes p_K (A_1 \times \dots \times A_K) \; \nu(dx_0)$$



]

If $p_{s,t} = \overline{p}_{t-s}$ for some family of trans prob $(\overline{p}_t)$ this is a Markov semigroup. $(p_{t+s} = p_t p_s)$

- An $\mathcal{F}_t$ adapted SP is a Markov process (of trans prob $(p_{s,t})$) if $\forall t \geq s$

$$\mathbb{P}(X_t \in B \mid \mathcal{F}_s) = p_{s,t}(X_s, B) \quad \text{a.s.}$$

Homogeneous if $p_{s,t} = \overline{p}_{t-s}$ is a Markov semigroup. $\left[\mathbb{P}(X_t \in B \mid \mathcal{F}_s) = \mathbb{P}(X_t \in B \mid X_s) \right]$.