Lecture 8 Markov processes

$(X_t, \mathcal{F}_t)_{t\geq 0}$ $\qquad \mathbb{P}(X_t \in B \mid \mathcal{F}_s) = p_{s,t}(X_s, B)$

for $(p_{s,t})_{s\leq t}$ a family of consistent trans. prob.

if $\mathcal{F}_s$ not specified, $\mathcal{F}_s = \mathcal{F}_s^X$

Homogeneous if $p_{s,t} = p_{t-s}$ is a Markov semigroup

$t_1, t_2, \dots t_n$ $\qquad \nu$ prob meas on $(\mathcal{X}, \mathcal{G})$

$\nu \otimes p_{0,t_1} \otimes \dots \otimes p_{t_{n-1},t_n}$ prob distr on $(\mathcal{X}^{n+1}, \mathcal{G}^{n+1})$

By def $\quad f \in b\mathcal{G}, \quad s \leq t$

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \int_{\mathcal{X}} f(x)\, p_{s,t}(X_s, dx)$$

$$(p_{s,t} f)(x) = \int f(y)\, p_{s,t}(x, dy)$$

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = (p_{st} f)(X_s)$$

Map $p_{s,t} : b\mathcal{G} \to b\mathcal{G}$.

$t_0 \leq t_1 \leq \dots \leq t_n \qquad f_1, \dots f_n \in b\mathcal{G}$

$h_n : \mathcal{X}^{[0,\infty)} \longrightarrow \mathbb{R} \qquad h_n \in b\mathcal{G}^{[0,\infty)}$

$$h_n(x) = f_1(x_{t_1}) \cdot \dots \cdot f_n(x_{t_n})$$

$$\mathbb{E}[h_n(X) \mid \mathcal{F}_{t_0}] = \mathbb{E}[f_n(X_{t_n}) \cdots f_1(X_{t_1}) \mid \mathcal{F}_{t_0}]$$

$$= \mathbb{E}\left[\mathbb{E}[f_n(X_{t_n}) \mid \mathcal{F}_{t_{n-1}}]\, f_{n-1}(X_{t_{n-1}}) \cdots f_1(X_{t_1}) \mid \mathcal{F}_{t_0} \right]$$

$$= \mathbb{E}\left[(p_{t_{n-1},t_n} f_n)(X_{t_{n-1}})\, f_{n-1}(X_{t_{n-1}}) \cdots f_1(X_{t_1}) \mid \mathcal{F}_{t_0} \right]$$

$$= h_0(X_{t_0})$$

$$h_{\ell-1}(x) = f_{\ell-1}(x) \cdot (p_{t_{\ell-1} t_\ell} h_\ell)(x) \qquad f_0(x) = 1$$

$$\mathbb{E}[f_n(X_n) \cdots f_1(X_1) \mid \mathcal{F}_{t_0}] = p_{t_0 t_1}(f_1 \cdot p_{t_1 t_2}(f_2 \cdots \quad f_n)))(X_{t_0})$$

<u>Theorem</u> $(\mathcal{X}, \mathcal{G})$ $\mathcal{B}$ isomorphic. $(p_{st})_{s \le t}$ consistent ν prob measure on $(\mathcal{X}, \mathcal{G})$. Define fdd s

$$\mu_{0 t_1 \ldots t_n} = \nu \otimes p_{0 t_1} \otimes \cdots \otimes p_{t_{n-1}, t_n}$$

Then

(1) $\exists$ SP $(X_t, \mathcal{F}_t^X)$ having these fdd.

(2) (X_t) is a Markov process tr prob $p_{s,t}$.

(3) Any MP $(\tilde{X}_t)$ with tr prob $p_{s,t}$ and st $\mathbb{P}(\tilde{X}_0 \in B) = \nu$ $\forall B$ has same law as (X_t). $\square$

<u>Proof</u> (1) Canonical construction. Need to check consistent $(p_{st}) \Rightarrow$ consistent fdd.

$$\nu \otimes p_{0t_1} \otimes \cdots \otimes p_{t_{n-1} t_n}(A_0 \times \ldots \times \underset{\mathcal{X}}{\overset{}{A_k}} \times \ldots A_n) =$$

$= \nu \otimes \cdots \otimes p_{t_{k-1},t_{k+1}} \otimes \cdots \otimes p_{t_{n-1}t_n}(A_0 \times \cdots \times A_{k-1} \times A_{k+1} \times \cdots A_n)$

use $\quad p_{t_{k-1}t_k} p_{t_k t_{k+1}} = p_{t_{k-1}t_{k+1}}$.

(2) Need to check $\forall t \geq s$

$$\mathbb{P}(X_t \in B \mid \mathcal{F}_s^X) = p_{st}(X_s, B)$$

to do this show $\forall A \in \mathcal{G}^{[0,s]}$

$$\mathbb{P}(X_t \in B, X_{[0,s]} \in A) = \mathbb{E}\{\mathbb{I}_{X_{[0,s]} \in A}\, p_{st}(X_s, B)\}$$

Sufficient to check on

$A = \{x_{s_1} \in A_1 \ldots x_{s_k} \in A_k\}$ $\quad \forall s_1 \leq s_2 \ldots \leq s_k \leq s$, $A_1 \ldots A_k \in \mathcal{G}$

$$\mathbb{P}(X_t \in B\ X_{s_1} \in A_1 \ldots X_{s_k} \in A_k)$$
$$= \mathbb{E}[p_{st}(X_s, B)\, \mathbb{I}_{X_{s_1} \in A_1} \cdots \mathbb{I}_{X_{s_k} \in A_k}]$$

check equality by recursion formula.

(3) Any MP with $\mathbb{P}(\tilde{X}_0 \in B) = \nu(B)$ and trans prob (p_{st}) has same law as X_t ..

Indeed must have same fdd's (Recursion)
$\Rightarrow$ same law. $\square$

wts

$$\mathbb{P}(X \in A \mid \mathcal{G}) \overset{a.s.}{=} Y \quad \text{for } X \in m\mathcal{F}, \; Y \in m\mathcal{G}.$$

$$(\Omega, \mathcal{F}, \mathbb{P}), \quad \mathcal{G} \subseteq \mathcal{F}$$

Call $\quad Z = \mathbb{P}(X \in A \mid \mathcal{G})$

wts $\quad Z = Y$ a.s.

$$\mathbb{E}(Z \mathbb{I}_A) = \mathbb{E}(Y \mathbb{I}_A) \quad \forall A \in \mathcal{G} \Rightarrow Z = Y \text{ a.s.}$$

equivalently $\quad W \in m\mathcal{G} \quad W = Z - Y$

$$\mathbb{E}\, W \mathbb{I}_A = 0 \quad \forall A \in \mathcal{G} \Rightarrow \quad W = 0 \text{ a.s.}$$

$$A = \{\omega : W > 0\} \Rightarrow \mathbb{E}\, W_+ = 0.$$

$\mathbb{P}_\nu$ the law of MP X_t with trans prob $p_{s,t}$

$X_0 \sim \nu$. (unique on $(\mathcal{X}^{[0,\infty)}, \mathcal{G}^{[0,\infty)})$)

$\mathbb{P}_x = \mathbb{P}_{\delta_x}$ $\qquad x \in \mathcal{X}$

Lemma $\forall A \in \mathcal{G}^{[0,\infty)}$ $\quad x \mapsto \mathbb{P}_x(A)$

is measurable on $\mathcal{G}$. and.

$$\mathbb{P}_\nu = \int_{\mathcal{X}} \mathbb{P}_x \, \nu(dx) \qquad [\mathbb{P}_\nu(A) = \int_{\mathcal{X}} \mathbb{P}_x(A) \nu(dx)$$

Example $(X_t)_{t \geq 0}$ indep increments.

is Markov

$$\mathbb{P}(X_t \in B | \mathcal{F}_s^X) = \mathbb{P}(X_t - X_s \in B - X_s | \mathcal{F}_s^X]$$

$$= p_{s,t}(X_s, B)$$

$$p_{st}(x, B) = P_{X_t - X_s}(B - x)$$

- If X_t stationary indep increm.

$\Rightarrow$ homogeneous MP because

$P_{X_t - X_s}$ only depend on $t-s$.

eg $\quad W_t$ standard Wiener process

Homogeneous Markov.

$X_t = Z + at + W_t \qquad$ for Z indep of (W_t)

also Markov $\qquad X_t - X_s = a(t-s) + W_t - W_s.$

homog.

$$p_t(x,dy) = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y-x-vt)^2}{2t} \right\} dy \qquad \square$$

Def $\quad \mathbb{P}_x$ law $\underset{\text{homogeneous}}{\wedge}$ MP with semigroup (p_t)

prob measure ν on $(\mathcal{X}, \mathcal{G})$ is invariant for p_t if

$$\mathbb{P}_\nu = \int \mathbb{P}_x \, \nu(dx) \qquad \text{is invariant under shift}$$

$$\mathbb{P}_\nu(A) = \mathbb{P}_\nu(\theta_s^{-1}(A)) \qquad \square$$

Lemma $\quad \nu$ invariant (p_t) iff.

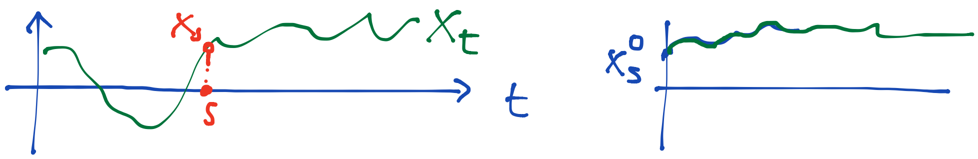
$$\nu \otimes p_t(\mathcal{X} \times B) = \nu(B) \qquad \forall B \in \mathcal{G},\ t \geq 0.$$

$$\left[\text{ie} \quad \int_{\mathcal{X}} p_t(x,B)\, \nu(dx) = \nu(B) \right] \qquad \square$$

Proposition [Markov property] $(P_t)_{t\geq 0}$ Markov semigroup on $(\mathcal{X}, \mathcal{G})$. $\mathbb{P}_x$ law of $(X_t)_{t\geq 0}$ homog MP started at x. $h \in b\mathcal{G}^{[0,\infty)}$

Then (1) $x \mapsto \mathbb{E}_x h(X)$ is in $m\mathcal{G}$.

(2) $\mathbb{E}[h(\theta_s X) \mid \mathcal{F}_s] = \mathbb{E}_{X_s} h(X)$ $\square$



Proof Prove (1) and (2) for

$$h(x) = f_1(x_{t_1}) \cdot \ldots \cdot f_n(x_{t_n}) \qquad 0 \leq t_1 \leq \ldots \leq t_n$$
$$f_1 \ldots f_n \in b\mathcal{G}$$

and extend to general $h \in b\mathcal{G}^{[0,\infty)}$ by monotone class. (linearity of exp + bdd conv.)

$$h(\theta_s x) = f_1(x_{s+t_1}) \cdot \ldots \cdot f_n(x_{s+t_n})$$

$$\mathbb{E}[h(\theta_s X) \mid \mathcal{F}_s] = \mathbb{E}[f_n(X_{s+t_n}) \cdot \ldots \cdot f_1(X_{s+t_1}) \mid \mathcal{F}_s]$$
$$= h_0(X_s)$$

$$\begin{cases} h_n(x) = f_n(x), \quad h_{e-1}(x) = f_{e-1}(x) \cdot (p_{\underbrace{t_{e-1}+s,\, t_e+s}_{P_{t_e - t_{e-1}}}} h_e)(x) \\ h_{e-1}(x) = f_{e-1}(x) (p_{t_e - t_{e-1}} h_e)(x). \end{cases}$$

(rhs $\mathbb{E}_{X_s} h(X)$)

$$\mathbb{E}_x h(X) = \mathbb{E}_x [f_n(X_{t_n}) \cdots f_1(X_{t_1})] = h_0(x)$$

$$\Rightarrow \mathbb{E}_{X_s} h(X) = h_0(X_s) \text{ equal to lhs.}$$ $\square$

$\mathbb{P}_x = \mathbb{P}_{\delta_x}$ prob measure on $(\mathcal{X}^{[0,\infty)}, \mathcal{G}^{[0,\infty)})$

$\mathbb{P}_x(\mathcal{X}^{[0,\infty)}) = 1$.

$$\mathbb{P}_x(\{\omega : \omega(t_1) \in A_1, \ldots \omega(t_n) \in A_n\})$$

$$= (p_{0t_1}(x, \cdot) \otimes \ldots \otimes p_{t_{n-1},t_n})(A_1 \times \ldots \times A_n)$$

$$\mathbb{P}_x(\{\omega : \omega(t) \in A\}) = p_{0,t}(x, A).$$

$p_{s,t}$ consistent trans prob.

$p_{s,t} = p_{t-s}$. (p_t)

$$p_t : \mathcal{X} \times \mathcal{G} \to [0,1]$$
$$(x, B) \longmapsto p_t(x, B)$$

$p_t p_s = p_{t+s}$ [i.e. $p_{t+s}(x,B) = \int p_s(y,B)\, p_t(x,dy)$].

$p_0 p_t = p_t$.

$(p_t)_{t \geq 0}$ $(p_t, p_s) \longmapsto p_t \cdot p_s = p_{t+s}$.