Economic Policy Uncertainty & Asset Price Volatility

Maxim Ulrich

Columbia University
Economic Policy Uncertainty & Asset Price Volatility

Abstract

We document that fear about misspecified economic and central bank policies explain 45% of variations in bond option implied volatilities and interest rate volatilities. We endogenize this empirical pattern with a parsimonious equilibrium asset pricing model. In equilibrium, volatility is endogenously driven by fear of not knowing the data generating process that drives future economic and future central bank policies. An increase in either of these two uncertainties steepens the yield curve and increases the volatility in asset and option markets. A structural estimation of the equilibrium model explains the upward sloping term structures of interest rates, bond volatility, and option volatility, with only four in real-time observable economic and central bank related risk and uncertainty factors. The study ends with highlighting an inverse relationship between interest rates and volatility disparity from fundamentals during the policy hiking period of 2004-2007 and during QE1.
1 Introduction

The Chairman of the Federal Reserve emphasizes in Bernanke and Kuttner (2005) the importance of understanding how monetary policy risks transmit to asset markets. Surprisingly, little is known about how monetary policy and economic risks affect bond option markets and the volatility of interest rates. Notice the strong co-movement of different standardized volatility measures from Treasury bond and bond option markets in Figure 2. In fact, more than 90% of variations in these volatilities are driven by one unobserved factor. Can observable measures of economic and monetary policy risks account for at least a fraction of these variations? Is there a simple and parsimonious asset pricing model that explains the transmission mechanism from macroeconomic and monetary policy risks to the volatility of bonds and options? It is the goal of this paper to answer these questions.

Guided by existing state of the art asset pricing theories, we divide economic and monetary policy (inflation) risks into three subgroups: business cycle risk (‘Activity’), volatility risk (‘Risk’), and model risk (‘Uncertainty’). We document in Table 2 that (i) 50% of variations in the above mentioned unobserved bond volatility factor are explainable by observable macroeconomic risk factors, (iii) variations in Activity and Risk seem not to drive the volatility in the bond market, (iii) variations in Uncertainty soak up the entire 50% of explanatory power, (iv) controlling for the yield curve does not drive out Uncertainty as an important contributor to bond volatility, and (v) economic and monetary policy Activity and Uncertainty are key macroeconomic risk factors for variations in Treasury interest rates.

It is unfortunate for economic research in general and for the understanding of the monetary policy transmission mechanism in particular, that existing asset pricing theories cannot explain why Economic Uncertainty and Monetary Policy Uncertainty are such important volatility risk factors in the bond and in the option market. We close this gap in the literature by proposing a parsimonious equilibrium asset pricing model where asset price volatility is endogenously driven by Economic Uncertainty and Monetary Policy Uncertainty.

The agent in our model is exposed to both, Knightian uncertainty about future Economic Activity and Knightian uncertainty about future Monetary Policy Activity. Notice that com-

---

1 Bernanke and Kuttner (2005) write in their introduction: ”Understanding the links between monetary policy and asset prices is thus crucially important for understanding the policy transmission mechanism.”

2 Monetary policy risks and uncertainties relate to inflation because one of the key Federal Reserve mandates, given by U.S. Congress, is to keep inflation under control. Even unconventional Quantitative Easing policies of central banks around the globe were motivated as necessary tools to ensure stable inflation in the future.
pared to Risk, the investor is not afraid of not knowing the realized values of future Economic Activity and future Monetary Policy Activity ex-ante, but instead he fears not to know their respective data generating process (DGP). The time-varying magnitude of Economic Uncertainty and Monetary Policy Uncertainty is exogenous and heteroscedastic and leads in equilibrium to a heteroscedastic dynamic of marginal utility, interest rate volatility, and to stochastic option implied volatilities. The model accounts very naturally for the high co-movement between interest rate volatility and option implied volatility that is revealed in Figure 2.

Our goal is not to understand the reason for Uncertainty but its equilibrium effect onto the volatility of interest rates and bond options. Our equilibrium model contributes several theoretical insights to the existing literature, from which we emphasize two. First, the source of Uncertainty matters. An exogenous increase in Economic Uncertainty leads to a flight to safety and a steepening of the yield curve because being long a Treasury bond hedges the amplified amount of Economic Uncertainty. On the other hand, holding a Treasury bond does not hedge run-ups in Monetary Policy Uncertainty. Second, an increase in Economic Uncertainty and/or Monetary Policy Uncertainty leads to an endogenous increase in the volatility of interest rates and in bond option implied volatilities. The relative persistence of both Uncertainty measures and the relative strength of the agent’s fear determine whether Economic Uncertainty or Monetary Policy Uncertainty dominates the endogenous dynamic of volatility in the bond market. Whereas the persistence of both Uncertainty measures is pinned down by their respective time series behavior, we have to estimate the equilibrium model to assess the agent’s degree of fear.

We estimate the structural model with Maximum Likelihood and a rich panel of interest rate, macroeconomic, monetary policy, interest rate volatility, and bond option implied volatility data. From the numerous empirical lessons, we want to stress six novel and policy relevant insights.

First, the parsimonious equilibrium model is able to explain, the upward sloping term structures of interest rate volatility, option implied volatility, and interest rates. Second, fear (i.e. Uncertainty) in the model is driven by two components, a high frequency component (i.e. Economic Uncertainty) and a low frequency component (i.e. Monetary Policy Uncertainty). Third, a potentially alarming finding for policy makers and central bankers is that the estimated half-life of a shock to Monetary Policy Uncertainty is over 50 years. Fourth, the parsimonious model captures that physical volatilities and option implied volatilities sharply increase before recessions, which is mainly driven by an increase in Uncertainty about future Monetary Policy Activity.
Fifth, according to our model, there was a significant underpricing of bond volatility during 2004 and 2007. This deviation from fundamentals was not caused by an artificially low market price of volatility, but instead by an understatement of physical bond volatility. Sixth, the estimated structural model reveals a striking negative correlation between Treasury yields starting to revert back to fundamentals in 2004, and bond volatility starting to negatively deviate from fundamentals. The Federal Reserve started to raise interest rates in 2004 to correct the path of easy monetary policy that it launched right before the 2001 recession. The pricing errors of our model suggest that the bond market might have interpreted that as a signal towards lower financial market risk ahead, despite the amount of fundamental uncertainty in the macroeconomy (i.e. Monetary Policy Uncertainty) being unchanged. With the launch of untested QE1 policies (Quantitative Easing), bond interest rates started to trend significantly lower than predicted by fundamentals in our model, whereas bond volatilities started to move above fundamentals.

The theoretical results of our paper contribute to a very small but fast growing part of the literature that links volatility in financial markets to Knightian uncertainty about macroeconomic fundamentals.\(^3\) Liu et al. (2005) analyze how Knightian uncertainty about rare events affects equity options. The authors endogenize an uncertainty premium for exposure to rare events and show in a calibration exercise its importance for the volatility of equity options. Drechsler (2012) endows an extended Bansal and Yaron (2004) economy with a representative agent who is exposed to Knightian uncertainty about the frequency and magnitude of rare jumps in endowment growth, cashflow growth, and their volatility. Holding a financial option provides a natural hedge against these model misspecification concerns which, as a calibration shows, can explain the high variance risk premium in S&P500 index options. Miao et al. (2012) extend the smooth ambiguity model of Ju and Miao (2012) to account for the variance risk premium in equity options. The authors show that ambiguity aversion enlarges the counter-cyclical nature of the stock return variance and amplifies the variance risk premium in equity options.

These pioneering theoretical contributions study the effect of Knightian uncertainty on the variance risk premium in equity options and calibrate their respective model to several unconditional financial and macroeconomic moments. Unlike these papers, our theory shuts off both

---

\(^3\) Theoretical contributions to link financial market volatility to Knightian uncertainty are rare, but the foundations of the importance of Knightian uncertainty in financial economics is very rich. Early pioneers are Knight (1921), Ellsberg (1961), Gilboa and Schmeidler (1989), Chen and Epstein (2002), Anderson, Hansen, and Sargent (2003), Epstein and Schneider (2003), Uppal and Wang (2003), Klibanoff et al. (2005), Hansen et al. (2006), Hansen and Sargent (2008). Applications to financial economics are Epstein and Miao (2003), Maenhout (2004, 2006), Sbuelz and Trojani (2008), Gagliardini et al. (2009), Ju and Miao (2012), among others.
the rare events and/or stochastic volatility channel in fundamentals, the variance risk premium, and the dependence on unobserved risk factors. Instead, we explicitly analyze how Knightian uncertainty about future economic and monetary policy actions affects the volatility in the bond market. Another distinguishing feature of our modeling is the strong empirical focus of our analysis. We focus explicitly on the time series properties of bond volatility and analyze whether different types of observable macro and policy related risks and uncertainties account for these. Our equilibrium model is more parsimonious than the existing theories and allows for a straightforward maximum likelihood estimation with only observable macroeconomic risk variables. The empirical part of our paper is the first to document that an equilibrium model with Knightian uncertainty can indeed explain key features of interest rates, interest rate volatility, and bond option implied volatilities.

Other equilibrium asset pricing theories with Knightian uncertainty that are tested with empirical estimations are rare in the literature. Ulrich (2012) is the first study to document that an equilibrium model with an observable amount of Knightian uncertainty about future monetary policy actions is indeed able to explain a longstanding puzzle that the same model without Knightian uncertainty could not resolve. Ulrich (2012) focuses on the positive term premium in U.S. government bonds and can explain why the yield curve was rather flat in the early 1980s, whereas observable forecasts of inflation were strongly downward sloping. He shows that the observable amount of inflation uncertainty, measured as dispersion in professional macroeconomic forecasts, was strongly elevated during that time period. The ambiguity averse representative investor did therefore price Treasury bonds with a steep inflation ambiguity premium. While Wright (2011) provides international empirical evidence for the importance of this channel, Ulrich (2012) cannot explain why Knightian uncertainty might affect the conditional volatility of interest rates. Furthermore, his empirical implementation, following the tradition in the literature, relies on unobserved factors, which complicates policy relevant interpretations. Our study has a different focus because we focus on endogenizing volatility in the bond and option market with only observable measures of economic and monetary policy related Knightian uncertainty.

this exciting avenue of research, we show that a simple economy with Knightian uncertainty and only four observable state variables is able to make a step towards linking bond and option volatility to uncertainty about monetary policy. We leave a related analysis within a Bayesian learning framework to future research.

The rest of the paper is organized as follows. Section 2 is devoted towards understanding the empirical linkages between different types of macroeconomic risks and the volatility on bond and option markets. Section 3 proposes an equilibrium model that endogenizes key features of the lessons from Section 2. Section 4 presents the findings of the structural model estimation. Section 5 concludes and all proofs, description of data, as well as the specification of the likelihood function are summarized in the appendix.

2 Economic Risks and Bond Volatility

In this section we document that a significant fraction of volatility in Treasury bond and Treasury bond option markets correlates with variations in observable measures of economic risks. We differentiate between economic risks and risks associated with monetary policy. Monetary policy risks are related to inflation because first, the Federal Reserve controls the aggregate supply of money in the U.S. economy (and therefore inflation) and second, one of its mandate, given by U.S. Congress, is to ensure low rates of inflation. We classify economic risks to be related to real economic growth in the economy. The analysis will decompose both economic and monetary policy risks into three sub-categories each: Activity, Risk, and Uncertainty, which, as we show below, builds naturally on state of the art economic theory and macroeconomic data.

2.1 Different Economic Risk Measures

Our primary focus of the empirical analysis is to understand whether economic and policy risks could be responsible for variations in the volatility of bond markets. Guided by theory we differentiate between business cycle risks, volatility risks (which assumes the data generating process is known), and Knightian uncertainties. The seminal work of Knight (1921) emphasized the importance of distinguishing between knowing the DGP but not knowing the outcome ex-ante (volatility risk) from not knowing the DGP that the random variable is drawn from (Knightian how learning affects the time-varying correlation between stocks and bonds, whereas David and Veronesi (2012) analyze how learning about the Taylor rule affects the price of equity options.
uncertainty). The more recent seminal workhorse model of Bansal and Yaron (2004) emphasizes that innovations to business cycle variables can be important macroeconomic risk factors. We call such risk factors Activity as by their nature, they capture economic and policy activity.

We use real-time data from the Survey of Professional Forecasters to account for all three subcategories of economic and policy risks. We define the demeaned one quarter ahead median GDP growth forecast to coincide with Economic Activity. Similarly, we define the demeaned one quarter ahead median inflation forecast to coincide with Monetary Policy Activity. In order to have a standard measure for economic and monetary policy risks, we separately extract the conditional volatility of Economic Activity and Monetary Policy Activity with a GARCH(1,1). We call the former Economic Risk and the latter Monetary Policy Risk, as the literature associates risk usually with volatility. As has been argued elsewhere (e.g. Anderson et al. (2009), Patton and Timmerman (2010)), the amount of macroeconomic Knightian uncertainty can be naturally estimated with cross-sectional dispersion data of real-time macroeconomic forecasts. We measure Economic Uncertainty as the cross-sectional dispersion in one quarter ahead GDP growth forecasts, and we measure Monetary Policy Uncertainty as the cross-sectional dispersion in one quarter ahead inflation forecasts. We remove seasonality in both uncertainty measures by using a four quarter moving average.

Figure 1 plots all six economic risk factors for the time period 1994:Q1 to 2009:Q2 together with NBER recession dates. The data begins in 1994:Q1 because we only have access to bond volatility data starting in 1994:Q1. All risk factors in Figure 1 are standardized to have a zero mean and a unit variance. The top panel of Figure 1 plots Economic Activity and Monetary Policy Activity, the middle panel plots Economic Risk and Monetary Policy Risk, whereas the lower panel plots Economic Uncertainty and Monetary Policy Uncertainty.

Figure 1 provides graphical evidence for the time varying nature of all six types of economic risks and for the countercyclical nature of Risk and Uncertainty. Monetary Policy Activity fell during the 1990s and remained stable during the 2000s, which implies a reduction in long-run monetary policy risk (Piazzesi and Schneider 2006, Piazzesi and Schneider, 2010)). Economic Activity has been particularly strong in 2004 and 2006 and tends to be very weak during recessions, which implies amplified long-run economic risks during recessions (Bansal and Yaron, 2004). The middle panel of Figure 1 highlights that Monetary Policy Risk has been low and calm over the past 15 years with a few sudden spikes during recessions. While variations in

---

5 Compare for example Drechsler and Yaron (2011), Bansal et al. (2012), Campbell et al. (2012), Tauchen (2005), Bloom (2009), Bollerslev, Tauchen, and Zhou (2009).
Economic Risk are more volatile, the pronounced spikes during recessions coincide with spikes in Monetary Policy Risk.

Figure 1 also reveals that both Economic Uncertainty and Monetary Policy Uncertainty have been low during the 1990s but increased considerably after the 2001 recession. Whereas both measures of uncertainty move similarly, a noticeable divergence exists for the period 2003 to 2007. Monetary Policy Uncertainty fell substantially during this four year period, while Economic Uncertainty remained flat with a steady uprise that started in 2006. In the second half of 2007, Monetary Policy Uncertainty started to jump up. Economic Uncertainty followed this rapid increase with a roughly six month time lag.

2.2 Different Measures for Bond Volatility

To characterize the volatility in bond markets, the finance literature has generally taken two approaches: (i) extract volatilities from options written on Treasury bonds, and (ii) extract realized volatilities from squared bond returns. We employ both of these approaches here. Figure 2 plots two volatility measures for each of the two approaches. Notice that each volatility measure is standardized to have a zero mean and a unit variance. The option implied volatility measure is from Bloomberg and coincides with the 'TIV' data. The TIV is constructed in a similar way than the CBOE VIX. It measures the implied volatility of a one month option written on a Treasury note futures. $TIV(5\text{-}yr)$ ($TIV(10\text{-}yr)$) denotes the implied volatility for a one month option on entering a Futures contract on a Treasury note with a 6% coupon and a maturity of 5-years (10-years). In practice, Treasury futures allow the delivery of any coupon Treasury bond within a specified maturity range. A conversion factor is used to determine the price of the delivered security, assuming it would yield 6%. For the 5-year (10-year) Treasury futures, Treasury bonds with maturity of not more than 5 years and 3 months (ten years) and at least 4 years and 2 months (6.5 years) are deliverable.

Figure 2 also presents two volatility measures that we extract from bond return data. In order to determine the realized volatility of a Treasury bond we use daily zero-coupon Treasury yields from the Gurkaynak et al. (2007) database. We convert these yields into daily prices. Using these daily prices allows us to determine the average squared logarithmic return within one quarter to proxy for the realized quarterly variance of Treasury zero-coupon bonds. Note that it is well known that zero-coupon bonds are exposed to more interest rate risk than a 6% coupon bond with the same maturity. In order to address this issue we will compare the $TIV(5\text{-}yr)$ ($TIV(10\text{-}yr)$ with the realized volatility of a zero-coupon bond that has the same Macaulay
duration. This ensures that the 6% coupon bond that underlies the option contract has the same effective maturity than the zero-coupon bond that is used to determine the realized volatility. The appendix explains how we have calculated the Macaulay duration of the 6% coupon bond. For simplicity, we will refer in the subsequent empirical analysis to $RVol(5\text{-yr})$ ($RVol(10\text{-yr})$) as the realized volatility of a hypothetical 6% Treasury coupon bond with 5 (10) year maturity.

A closer look to Figure 2 reveals a striking co-movement between (i) volatilities of different maturities, as well as between (ii) option and Treasury bond implied volatilities of the same maturity. The strong co-movement of different bond volatility proxies suggests the existence of a single bond volatility factor. This bond volatility factor makes Treasury and option implied bond volatilities shoot up right before the start of recessions and exhibit a volatile behavior during recessions. Realized volatilities and option implied volatilities are strongly countercyclical, low in expansions and high in recessions.

2.3 Risks, Uncertainties and Bond Volatility: Empirical Assessments

Figure 1 revealed that Uncertainty shoots up before recessions, whereas Risk tends to shoot up during recessions. This pattern suggests that Uncertainty could be particularly important for the volatility in the bond market, which also tends to shoot up before recessions. Before we employ formal statistical tools, we graphically inspect in Figure 3 and in Table 1 the co-movement of bond volatility with aggregate Uncertainty and aggregate Risk in the U.S. economy.

We define the bond volatility factor, BondVol, to coincide with the first principal component of all four bond volatility proxies. The factor BondVol alone explains more than 90% of variations in volatilities of bonds and options. To aggregate economic and monetary policy information into one aggregate measure, we determine separately the first principal component of the economic and monetary policy Activity, Risk, and Uncertainty measures, respectively. This provides us with an economy-wide measure for Activity, Risk, and Uncertainty in the U.S. economy. Table 1 summarizes the cross-correlations of these aggregate risk measures.

The unconditional correlations from Table 1 reveal a striking correlation of 0.7076 between BondVol and Uncertainty. The unconditional correlation of BondVol with the more traditional Activity and Risk measures is only half as large. Of course, there is a significant correlation among all economic risk factors. In particular, aggregate Activity has a 0.57 (0.45) correlation coefficient with aggregate Uncertainty (Risk), whereas Risk and Uncertainty themselves have a correlation coefficient of 0.56. The lower panel of Table 1 reveals that Uncertainty could not only be an important component of BondVol but could also be responsible for variations in
the level and slope of the Treasury yield curve. An increase in BondVol, Risk, or Uncertainty tends to lower the level of the Treasury yield curve. This is consistent with an increased precautionary savings motive and a more pessimistic outlook about future growth, respectively. From all macroeconomic risk factors, it is aggregate Uncertainty that correlates strongest with the level and the slope of the Treasury yield curve, -0.65 and 0.64, respectively. The positive correlation between the slope of the Treasury yield curve and Uncertainty is consistent with findings in Wright (2011) and Ulrich (2012).

To visualize the stark co-movement between BondVol and aggregate Risk and Uncertainty further, we plot these three standardized time series in Figure 3. The strong co-movement of all three factors is clearly visible. Uncertainty seems to have a more direct effect on BondVol for the following three reasons: (i) aggregate Risk remained flat in 1994-1998, whereas aggregate Uncertainty and BondVol declined sharply, (ii) aggregate Uncertainty and BondVol started to spike already several months before the last two recessions, whereas aggregate Risk followed only with a considerable time lag, and (iii) Uncertainty picked up dramatically in 2005 and peaked in 2009, which was followed by a run-up in BondVol during 2006 and 2009. Notice that neither Risk nor Uncertainty can explain the sharp drop in bond volatility between 2005 and 2007, which could be evidence for a severe underpricing of risks and uncertainties in the two years preceding the great financial crisis. We will further explore this last observation in Section 4.2.

Next we run OLS regressions to identify whether Activity, Risk, Uncertainty, or any linear combination of these macroeconomic risk factors is helpful in explaining variations in BondVol. All subsequent regression results are summarized in Table 2.\textsuperscript{6}

As a first natural step we are interested in understanding whether a linear combination of Activity, Risk, and Uncertainty is helpful in explaining variations in BondVol, i.e.

\[ BondVol_t = \beta_1 \cdot \text{Activity}_t + \beta_2 \cdot \text{Risk}_t + \beta_3 \cdot \text{Uncertainty}_t + \epsilon_t. \]  

Column (I) in Table 2 reveals a truly surprising result, namely 49.75\% of variations in BondVol relate to a linear combination of aggregate Activity, Risk, and Uncertainty. Another surprising result is that only Uncertainty contributes significantly towards explaining BondVol. Its robust t-statistic is 6.72, whereas Risk and Activity have a t-statistic of -0.49 and -1.01, respectively.

Given the identified importance of Uncertainty, we wonder how much of the variations in

\textsuperscript{6} Notice that all variables are standardized to have a mean of zero and a unit variance.
BondVol can be explained by Uncertainty alone. We test this with the following regression

$$\text{BondVol}_t = \beta_1 \cdot \text{Uncertainty}_t + \epsilon_t.$$  \hfill (2)

Column (II) in Table 2 summarizes that aggregate Uncertainty alone explains a remarkable 50.08\% of variations in BondVol. A 1\% increase in the standardized measure of Uncertainty leads to an estimated 0.7\% increase in the standardized measure of BondVol.

As a second natural step we try to understand whether Uncertainty contains BondVol relevant information that is not embedded in the Treasury yield curve. Recent work in empirical term structure modeling highlights the importance of unspanned macroeconomic risk factors (Joslin et al., 2011, Buraschi and Whelan, 2012). To do so, we first want to understand how much of the variations in BondVol are attributable to variations in the Treasury yield curve. The cross-correlations in Table 1 suggest that there is potentially a substantial co-variation between the level and slope of the Treasury yield curve and BondVol. We formally test this by running the following regression,

$$\text{BondVol}_t = \beta_1 \cdot y_{L,t} + \beta_2 \cdot y_{S,t} + \beta_3 \cdot y_{C,t} + \epsilon_t,$$  \hfill (3)

were \(y_L, y_S, y_C\) correspond to the level, slope and curvature (first, second, and third principal component) factors that we extract from the Gurkaynak et al. (2007) Treasury yield database.\(^7\) It is known from Litterman and Scheinkman (1991) that level, slope, and curvature drive nearly all of the variations in the Treasury yield curve.

Column (III) of Table 2 reveals that variations in the yield curve explain 61.93\% of variations in BondVol. This confirms results in the finance literature that interest rate volatility is partly unspanned by interest rates.\(^8\) While level, slope, and curvature jointly explain roughly 60\% of variations in BondVol, the respective t-statistics suggest that the slope factor is of particular importance. We investigate this channel further by regressing BondVol onto the slope factor, i.e.

$$\text{BondVol}_t = \beta_1 \cdot y_{S,t} + \epsilon_t.$$  \hfill (4)

Column (IV) of Table 2 summarizes that the slope factor alone explains 53.46\% of variations in BondVol. We therefore end up with the finding that Uncertainty alone and the term spread in the yield curve (slope factor, \(y_S\)) explain each roughly half of the variations in BondVol. An immediate question arises, namely whether Uncertainty and \(y_S\) contain the same information.

\(^7\) Our panel of yields uses ten different Treasury yields with maturities ranging from one year to ten years.

\(^8\) Recent advances include Collin-Dufresne et al. (2009), Collin-Dufresne and Goldstein (2002), Andersen and Benzoni (2010).
or whether each of these factors carries its own BondVol relevant information. We approach this problem from two angels. First, we regress Uncertainty on \( y_S \) and jointly on \([y_L, y_S, y_C]\) to understand whether the yield curve spans all information embedded in Uncertainty. Second, we take the unspanned part of Uncertainty and ask whether it adds explanatory power to the regressions (4) and (3). The relevant regression output is summarized in Table 3.

With regard to the first analysis we find that 59% (48%) of variations in Uncertainty are unspanned by \( y_S \) ([\( y_L, y_S, y_C \]). The term spread \( y_S \) reveals the biggest amount of Uncertainty related information (Ulrich (2012), Wright (2011)). Roughly half of the variations in Uncertainty appear unspanned by the level, slope, and curvature. We denote Uncertainty\(\setminus y_S\) (Uncertainty\(\setminus\{y_L, y_S, y_C\}\)) to be the component of aggregate Uncertainty that is orthogonal to \( y_S \) ([\( y_L, y_S, y_C \]). To better understand the nature of Uncertainty and BondVol, we regress BondVol on yield curve information and Uncertainty related information that is orthogonal to the yield curve, i.e.

\[
\text{BondVol}_t = \beta_1 \cdot y_{S,t} + \beta_2 \cdot \text{Uncertainty}_{t\setminus y_S} + \epsilon_t
\]

(5)

\[
\text{BondVol}_t = \beta_1 \cdot y_{L,t} + \beta_2 \cdot y_{S,t} + \beta_3 \cdot y_{C,t} + \beta_4 \cdot \text{Uncertainty}_{t\setminus\{y_L, y_S, y_C\}} + \epsilon_t.
\]

(6)

Table 4 summarizes the regression output and reveals that Uncertainty is not driven out by yield curve information. All three principal components of the yield curve, as well as the unspanned part of aggregate Uncertainty are significant contributors to BondVol. The term spread and the unspanned part of uncertainty explain 62.57% of variations in BondVol, whereas level, slope, and curvature together with the unspanned part of Uncertainty explain 70.77% of variations in BondVol. We conclude that (i) Uncertainty is a very important macroeconomic risk factor for BondVol that appears to be partly unspanned by the yield curve, and (ii) after controlling for Uncertainty, traditional risk measures, such as Risk and Activity, do not help to explain variations in the volatility of the bond market. The unspanned nature of Uncertainty is consistent with Duffee (2011) and Joslin et al. (2011) who emphasize the importance of incorporating unspanned macro risks into fixed-income models to better capture bond premiums. Whereas the previous two papers focus on an empirical analysis and on bond premiums, we will show that a parsimonious equilibrium model with Knightian uncertainty can very naturally endogenize that Uncertainty should drive the volatility of interest rates.

We close the empirical examination with two last questions. First, which are the macroeconomic risk factors that even a parsimonious general equilibrium model should account for? Second, do we have to account for a variance risk premium (VRP) in bond options?
With regard to the first closing question, Table 2 and Table 5 provide a striking message, namely a parsimonious general equilibrium model for Treasury interest rates, interest rate volatilities, and Treasury options should incorporate economic and monetary policy Activity and Uncertainty measures, and ignore the Risk measures.

In particular, Table 2 revealed that Uncertainty explains roughly 50% of variations in the volatility of bond prices and bond options, whereas Risk and Activity do not help to explain variations in BondVol. A general equilibrium model derives also implications for the term structure of interest rates. It is well known that the level factor explains more than 90% of variations in the term structure of interest rates (Litterman and Scheinkman (1991)). Table 5 reveals that Activity and Uncertainty contribute significantly to variations in $y_L$, whereas Risk is negligible. Column (II) of Table 5 reveals that Economic Activity, Monetary Policy Activity, Economic Uncertainty, and Monetary Policy Uncertainty explain jointly roughly 70% of variations in $y_L$, whereas Economic Risk and Monetary Policy Risk have robust t-statistics of less than 0.5. Monetary Policy Uncertainty has the highest absolute value of the robust t-statistic (7.40), emphasizing its overall importance for the level (Table 5) and volatility (Table 2) of interest rates. Wright (2011) and Ulrich (2012) explore the importance for the level and slope of interest rates, whereas we analyze the importance for the volatility of interest rates and the implication for bond options.

With regard to our second closing question, we wonder whether a parsimonious general equilibrium model could abstract from the variance risk premium in bond options. The importance of the variance risk premium for predicting equity and Treasury excess returns is well established in the literature.\(^9\) The focus of our paper is not on using option implied information to predict excess returns but rather to endogenize the dynamic of BondVol with a parsimonious equilibrium model. We document in Table 1 that interest rate volatilities have nearly identical dynamics in the bond and in the option market. Intuitively, 91% of variations under the pricing and under the physical probability measure are identical. The spread between both measures, the VRP, accounts therefore only for a rather small portion of volatility. To aggregate information, we determine the aggregate VRP as the first principal component of the spread between the 5-year (and 10-year) $TIV^2$ and the corresponding realized (physical) variance. Table 6 summarizes that regressing the aggregate VRP measure jointly on Activity, Risk, and Uncertainty,\(^9\)

---

\(^9\)The importance of the variance risk premium for predicting equity returns has been documented in Bollerslev et al. (2009) and Drechsler and Yaron (2011). The importance for predicting bond returns has been documented in Mueller et al. (2011).
i.e.

\[ \text{VRP}_t = \beta_1 \cdot \text{Activity}_t + \beta_2 \cdot \text{Risk}_t + \beta_3 \cdot \text{Uncertainty}_t + \epsilon_t, \]

reveals that all macro and policy risks together explain only 5.51% of variations in the VRP of bond options. Only Risk is weakly significant with a robust t-statistic of -2.1, whereas the loadings on Activity and Uncertainty are not significant. The regression results are in line with the literature, such as Mueller et al. (2011). The authors show that only disagreement about prices of short-term bonds and disagreement about prices of long-term bonds, expressed as dispersion of short-term and long-term interest rates, help to explain variations in the VRP of bond options. The authors analog for our Risk and Uncertainty measures do not help to explain the dynamic of the VRP in a statistically significant way. Buraschi and Whelan (2012) endogenize disagreement about bond prices in a Bayesian model with heterogeneous agents and informative signals and emphasize that disagreement about prices cannot easily arise in a representative agent economy with Knightian uncertainty. Taking all of these observations together motivates us to abstract from the VRP in our general equilibrium analysis to keep the analysis parsimonious and to focus on first-order effects.

3 General Equilibrium Model with Endogenous Volatility

The findings in the previous section show that an increase in Uncertainty is strongly related to an increase in the volatility of interest rates and bond options. It is an unresolved issue to understand in a general equilibrium model why economic and monetary policy Uncertainty matter so strongly for the volatility of the bond market. Ulrich (2012) explained why Monetary Policy Uncertainty matters for the term spread but cannot explain its importance for the bond option market. Drechsler (2012) calibrates a long-run risk model with Economic Uncertainty to capture the variance risk premium in equity options, but does not address the issue of bond market volatility or of Monetary Policy Uncertainty. Moreover, Drechsler (2012) shows that equity options rely crucially on a variance risk premium, whereas we document that the variance risk premium is of secondary importance for the dynamic of bond options. Buraschi and Whelan (2012) and Xiong and Yan (2010) explain how Economic Uncertainty can affect interest rates and bond premia, but leave unexplored how Monetary Policy Uncertainty affects bond options.

We now investigate whether a parsimonious macro-finance general equilibrium model with Economic Activity, Monetary Policy Activity, Economic Uncertainty, and Monetary Policy Uncertainty is able to match key features of the volatility in bond markets. To do so, we follow the
Knightian uncertainty literature by first introducing the agent’s most trusted belief about the data DGP of Economic Activity and Monetary Policy Activity in the economy (benchmark model). In a second step we will introduce that the agent has time-varying model misspecification doubts about his most trusted benchmark model.

To simplify all mathematical derivations, we assume that (i) time is continuous, varying over \( t \in [0, \ldots, \infty) \), (ii) all Brownian motions are pairwise orthogonal, and (iii) the representative agent has logarithmic preferences. We follow a standard procedure by assuming a complete filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, Q^0) \), where \( Q^0 \) stands for the agent’s most trusted benchmark DGP. Expectations under \( Q^0 \) are denoted as \( E_{Q^0}[] \) instead of \( E^{Q^0}[] \). The analysis explains in detail how the solution to the dynamic Gilboa and Schmeidler (1989) type min-max problem determines endogenously the worst-case probability measure \( Q^b \).

### 3.1 Investor’s Most Trusted View on Monetary Policy and Real Growth

The goal of the model is to endogenize stochastic volatility in bond markets without relying on what we called Risk in Section 2. The model will therefore rely on an homoscedastic endowment economy with an agent who fears not to know the true DGP for Economic Activity and Monetary Policy Activity. We do not aim to uncover why the agent fears to be confronted by model misspecification doubts, but analyze its possible consequences for the volatility on bond markets.

We model monetary policy and the real economy as exogenous processes and leave it to future research to endogenize these processes. The agent believes that monetary policy controls the trend growth rate, \( w_t \in \mathcal{R} \), of inflation, \( d \ln p_t \in \mathcal{R} \). The agent understands that realized inflation, \( d \ln p_t \), is a noisy realization of the central bank’s intended rate of trend inflation, \( w_t \). Inspired by our previous empirical analysis we denote \( w_t \) to coincide with Monetary Policy Activity. The real economy is characterized through the agent’s exogenous consumption growth process, \( d \ln c_t \in \mathcal{R} \), whose trend growth rate, \( z_t \in \mathcal{R} \), is time-varying. In analogy to our previous empirical exercise we denote \( z_t \) to capture Economic Activity. The agent believes that Monetary Policy Activity can only in the short-run affect Economic Activity. We model this by allowing for a non-zero correlation between Monetary Policy Activity shocks and Economic Activity shocks. This non-zero correlation ensures long-run inflation neutrality, which is consistent with a vertical long-run Philips curve.

The agent’s most trusted DGP of monetary policy actions and real growth can be conve-
niently described as:
\[
\begin{pmatrix}
    d \ln c_t \\
    d \ln p_t
\end{pmatrix}
= \begin{pmatrix}
    c_0 + z_t \\
    p_0 + w_t
\end{pmatrix}
\begin{bmatrix}
    0 & \sigma_c \\
    0 & \sigma_p
\end{bmatrix}
\begin{pmatrix}
    dW^c_t \\
    dW^p_t
\end{pmatrix},
\] (8)
where \([c_0, p_0, \sigma_c, \sigma_p] \in \mathbb{R}^4_+\). Economic Activity and Monetary Policy Activity follow a zero-mean Gaussian process
\[
\begin{pmatrix}
    dz_t \\
    dw_t
\end{pmatrix}
= \begin{pmatrix}
    \kappa_z z_t \\
    \kappa_w w_t
\end{pmatrix}
\begin{bmatrix}
    \sigma_{1z} & \sigma_{2z} \\
    0 & \sigma_w
\end{bmatrix}
\begin{pmatrix}
    dW^r_t \\
    dW^{MP}_t
\end{pmatrix},
\] (9)
with \([-\kappa_z, -\kappa_w, \sigma_{1z}, \sigma_w, -\sigma_{2z}] \in \mathbb{R}^5_+\). The modeling of \(\sigma_{2z} < 0\) is consistent with recent empirical macro-finance evidence in Piazzesi and Schneider (2006, 2010) and Ulrich (2012).

The investor has log utility with \(\rho\) characterizing his subjective time discount factor, i.e.
\[
U(c_t; z_t) = E_t \left[ \int_t^\infty e^{-\rho s} \ln c_s ds \right].
\] (10)

### 3.2 Fear about DGP Misspecifications

The agent worries that the \(Q^0\) DGP for Economic Activity and Monetary Policy Activity in equation (9) is not correct. The true but unknown one-step ahead DGPs might differ by statistically difficult to identify stochastic perturbations \(\{h^r_t, h^{MP}_t\}_t\). Each stochastic perturbation induces a conditionally Gaussian \(Q^h\) DGP for Economic Activity and Monetary Policy Activity, i.e.
\[
\begin{pmatrix}
    dz_t \\
    dw_t
\end{pmatrix}
= \begin{pmatrix}
    \kappa_z z_t \\
    \kappa_w w_t
\end{pmatrix}
\begin{bmatrix}
    \sigma_{1z} & \sigma_{2z} \\
    0 & \sigma_w
\end{bmatrix}
\begin{pmatrix}
    dW^{r,h}_t + h^r_t dt \\
    dW^{MP,h}_t + h^{MP}_t dt
\end{pmatrix}.
\] (11)
Notice that knowing the set of potentially correct one-step ahead perturbations \(\{h^r_t dt, h^{MP}_t dt\}_t\) is equivalent to knowing the set of potentially correct DGPs for Economic Activity and Monetary Policy Activity.\(^{10}\)

The agent observes central bank actions, which allows him to observe innovations in Monetary Policy Activities, \(dW^{MP}_t\). The policy relevant trouble with model misspecification doubts is that the agent worries that \(dW^{MP}_t\) was not drawn from a mean zero Gaussian distribution, but might indeed have come from a slightly perturbed density function. The magnitude of the conditional one-step ahead perturbation is quantified through \(h^{MP}_t\). Mathematically, this means that the agent observes only
\[
\frac{1}{\sigma_w} (dw_t - \kappa_w w_t dt) = dW^{MP,h}_t dt + h^{MP}_t dt,
\] (12)
\(^{10}\) A formal proof is given in Chen and Epstein (2002).
where we will solve for $h_t^{MP} \, dt$ as part of the min-max optimization problem below.

How is a rational agent going to cope with Knightian uncertainty about Monetary Policy Activities? The answer is surprisingly intuitive, as he will surveil with highest scrutiny whether there is statistical evidence that realized Monetary Policy Activity innovations were not drawn from $Q^0$ but instead from a perturbed density $Q^h$. Mathematically, this allows for a convenient characterization because the logarithmic likelihood ratio between a perturbed Gaussian one-step ahead DGP and the most trusted (Gaussian) DGP for Monetary Policy Activity innovations, $dW^{MP}$, coincides with

$$
\ln LR_{t \rightarrow t+dt}^{MP} := \ln \frac{Q_t^{h, W^{MP}}}{Q_0^{h, W^{MP}}} = -\frac{1}{2}(h_t^{MP})^2 dt + h_t^{MP} dW_t^{MP},
$$

(13)

where $\ln LR_{t \rightarrow t+dt}^{MP}$ measures the amount of statistical (empirical) confidence that Monetary Policy Activity is accurately described by a zero-mean Gaussian process.

The endogenous amount of model mistrust about future Monetary Policy Activities, $\ln LR^{MP}$, is an exponential martingale with stochastic volatility $h_t^{MP}$. This has the intuitive consequence that the agent expects incoming Monetary Policy Activity innovations to confirm that the most trusted DGP for Monetary Policy Activity ($Q^0$) is indeed correct, i.e. $E_t[dQ_t^{0, W^{MP}}] > E_t[dQ_t^{h, W^{MP}}]$. If the set of potentially correct DGPs for future Monetary Policy Activities increases ($h^{MP}$ increases), the endogenous heteroscedasticity in the agent’s confidence in $Q^0$ will increase. This implies that for each unit of unpredictable innovation in Monetary Policy Activity ($dW^{MP}$), the agent’s confidence in $Q^0$ will vary more strongly over time. We show below that this translates into stronger variations in the agent’s marginal utility per unit exposure to $dW^{MP}$.

After interpreting incoming Monetary Policy Activity innovations, the agent is able to infer realized innovations in Economic Activity ($dW^r$). For completeness, we also allow the agent to have model misspecification doubts about the DGP of $dW^r$. Mathematically, this means

$$
\frac{1}{\sigma_{1z}} \left( dz_t - \kappa_z z_t - \sigma_{2z} (dW_t^{MP,h} + h_t^{MP} \, dt) \right) = dW_t^{r,h} + h_t^r \, dt
$$

(14)

is observed, but its decomposition is unobserved and the reason for Economic Uncertainty. Similar to Knightian uncertainty about Monetary Policy Activities, the agent tries to learn through likelihood ratio tests which DGP drives future Economic Activities. The logarithmic likelihood ratio between the perturbed and the most trusted DGP for future Economic Activities, $W_t^r$, coincides with

$$
\ln LR_{t \rightarrow t+dt}^r := \ln \frac{Q_t^{h, W^r}}{Q_0^{h, W^r}} = -\frac{1}{2}(h_t^r)^2 dt + h_t^r dW_t^r.
$$

(15)
We follow the research on dynamic recursive multiple priors, pioneered by Chen and Epstein (2002) and Epstein and Schneider (2003), and assume the agent observes the set of potentially correct economic and monetary policy one-step ahead perturbations \( \{ h^r_t dt, h^{MP}_t dt \} \). Both likelihood ratios, mentioned above, imply that observing the set of one-step ahead perturbations, characterizes the set of potentially correct DGPs for both, Economic Activity and Monetary Policy Activity. Mathematically, we constrain the amount of Economic Uncertainty and the amount of Monetary Policy Uncertainty with the concept of multi-dimensional stochastic \( \kappa \)-ignorance of Chen and Epstein (2002), i.e.

\[
\frac{1}{2} (h^i_t)^2 dt \leq \text{Uncert}^i_t dt \in \mathcal{R}^+, \forall i \in \{ r, MP \}.
\]  

(16)

Notice that the amount of both uncertainties is observable and time-varying. We introduce the time-variation by assuming that both uncertainties are driven by their own exogenous processes that we denote as \( \eta^r_t \) and \( \eta^{MP}_t \), respectively. Formally,

\[
\text{Uncert}^i_t dt := \frac{(m^i)^2}{2} (\eta^i_t)^2 dt,
\]  

(17)

\[
d\eta^i_t = (a^i + \kappa^i \eta^i_t) dt + \sigma^i \sqrt{\eta^i_t} dW^i_t, \forall i \in \{ r, MP \},
\]  

(18)

with \([m^i, a^i, -\kappa^i, \sigma^i] \in \mathcal{R}^4, \forall i \in \{ r, MP \}\). Note that the scaling parameters \( m^i \forall i \in \{ r, MP \} \) govern the absolute magnitude of \( \text{Uncert}^i_t \), whereas the time series behavior is governed by \( \eta^i_t \). The heteroscedasticity in the amount of Economic Uncertainty (\( \eta^r_t \)) and Monetary Policy Uncertainty (\( \eta^{MP}_t \)), makes the set of potentially correct DGPs heteroscedastic and will endogenously create stochastic volatility in the agent’s stochastic discount factor and in the equilibrium bond and option market.

### 3.3 Optimization Problem and Endogenous Characterization of \( Q^h \)

The investor has min-max preferences and searches within the set of potentially correct DGPs for the unique economic and monetary policy Activity DGP that minimizes his expected lifetime utility. Formally, the investor solves

\[
\min_{h \in \{ h^r_t, h^{MP}_t \} } E^h \left[ \int_t^{\infty} e^{-\rho(s-t)} \ln c_s ds \bigg| \mathcal{F}_t \right]
\]  

(19)

\[
s.t. (8), (11), (16), (17), (18).
\]  

(20)
The solution to the minimization problem follows Chen and Epstein (2002) and is summarized in the following proposition.\footnote{Other applications of Chen and Epstein (2002) include Sbuelz and Trojani (2002, 2008), Drechsler (2012), and Ulrich (2012), among others.}

**Proposition 1**
The data generating process for Economic Activity and Monetary Policy Activity that minimizes the agent’s expected life-time utility is characterized by the following one-step ahead perturbations

\[
\begin{align*}
    h_r(t) &= -\sqrt{2} \cdot \text{Uncert}_r^t \in \mathcal{R}^- \tag{21} \\
    h_{MP}^t &= \sqrt{2} \cdot \text{Uncert}_{MP}^t \in \mathcal{R}^+. \tag{22}
\end{align*}
\]

The proof of Proposition 1 is in the appendix.

Proposition 1 suggests that the ambiguity averse investor does portfolio and asset pricing decisions, assuming that Economic Activity and Monetary Policy Activity are affected by the worst-case one-step ahead perturbations, \( h_r^t, h_{MP}^t \). Notice that these perturbations could indeed characterize the true but unknown DGP of Economic Activity and Monetary Policy Activity. To save notation, we will from now on refer to \( h_r^t dt \) and \( h_{MP}^t dt \) as the optimally chosen one-step ahead equilibrium distortions.

Another intuitive insight from Proposition 1 is that periods of increased Monetary Policy Uncertainty (Economic Uncertainty) are periods where the representative agent’s subjective forecast of future Monetary Policy Activity (Economic Activity) deviates more strongly from the most trusted \( Q^0 \) benchmark forecast. Finally, the endogenously chosen (worst-case) one-step ahead belief perturbations \( h_r^t dt \) and \( h_{MP}^t dt \) for anticipated future economic and monetary policy Activity introduce stochastic volatility into the agent’s marginal rate of substitution.

### 3.4 Endogenous Marginal Rate of Substitution

The ambiguity averse investor evaluates his expected life-time utility under the worst-case DGP, i.e.

\[
U^h(c_0; z_0, \eta_0^r, \eta_0^{MP}) = E_0^h \left[ \int_0^\infty e^{-\rho t} \ln c_t dt \right], \tag{23}
\]
where $z_0, \eta_0^{r}, \eta_0^{MP}$ emphasize the dependence on the revealed amount of Knightian uncertainty. From an econometric point of view it is beneficial to re-write the agent’s expected life-time utility under $Q_0$, because $Q_0$ is the agent’s most trusted DGP for the economy and therefore the most likely description of how the exogenous processes of the economy move in the data.

Applying the Radon Nikodym derivative allows us to re-write the agent’s worst-case expected life-time utility under $Q_0$, i.e.

$$U^h(c_0; z_0, \eta_0^{r}, \eta_0^{MP}) = E_0 \left[ \int_0^\infty LR_{0\rightarrow t}^r \cdot LR_{0\rightarrow t}^{MP} \cdot e^{-\rho t} \ln c_t dt \right]$$  \hspace{1cm} (24)

$$\frac{d \left( LR_{t\rightarrow t+dt}^r \cdot LR_{t\rightarrow t+dt}^{MP} \right)}{LR_{t\rightarrow t+dt}^r \cdot LR_{t\rightarrow t+dt}^{MP}} = -\sqrt{2} \cdot \text{Uncert}_t^r \cdot dW_t^r + \sqrt{2} \cdot \text{Uncert}_t^{MP} \cdot dW_t^{MP}. \hspace{1cm} (25)$$

The product of both marginal likelihood ratios coincides with the accumulated amount of confidence in the accuracy of $Q_0$ for predicting future economic and policy actions. Hansen and Sargent (2008) recommend to interpret this confidence measure as an endogenously constrained preference shock. The endogenous preference shock has no equilibrium asset pricing effects if the investor has 100% confidence in $Q_0$. This occurs if realized data on Economic Activity and Monetary Policy Activity confirm that $Q_0$ is the single most reliable forecasting DGP (i.e. $\ln LR^r \cdot \ln LR^{MP} \Downarrow 0$). On the other hand, this endogenous preference shock has severe asset pricing implications in periods where the agent is confronted with sufficient doubt that $Q_0$ might be incorrect.

Suppose the agent observes an unpredictable reduction in Economic Activity (i.e. $dW_r^r < 0$). As an endogenous response, the investor’s doubt about the accuracy of the AR(1) assumption for Economic Activity increases, because the worst-case (but still potentially correct) DGP for Economic Activity predicted a negative innovation in $dW_r^r$. This is intuitive because since the DGP for Economic Activity is not known, a realized value for Economic Activity that is lower than forecasted under the AR(1) specification ($Q_0$), i.e. $z_{t+dt} < E_t[dz_t]$, increases the observed statistical evidence that the worst-case DGP might indeed be correct. News about lower than expected Economic Activity hurts a min-max preference investor twice. First, the investor is in a state of low consumption growth and second, the investor is confronted with more statistical evidence that his trusted $Q_0$ DGP for Economic Activity is too optimistic.

Conversely, suppose the agent observes an unpredictable increase in Monetary Policy Activity (trend inflation), i.e. $dW_{MP}^r > 0$. In equilibrium, this increases the agent’s fear that his trusted $Q_0$ forecast for future Monetary Policy Activity might be systematically downward biased. Because of $\sigma_{2z} < 0$ this implies that forecasts for Economic Activity might be too optimistic under $Q_0$. 

19
The endogenous intertemporal marginal rate of substitution (MRS), $m$, accounts for the standard logarithmic utility consumption risk kernel and for model misspecification doubts, which are driven by shocks to Economic Activity and shocks to Monetary Policy Activity. Note that it is somewhat surprising that although $m$ is an endogenous equilibrium outcome of the dynamic min-max optimization problem in Proposition 1, its equilibrium characterization is quite tractable, i.e.

$$m_{t,t+\Delta} = e^{-\rho\Delta} \cdot \left( \frac{c_{t+\Delta}}{c_t} \right)^{-1} \cdot LR_{t\rightarrow t+\Delta}^r \cdot LR_{t\rightarrow t+\Delta}^{MP}. \quad (26)$$

The equilibrium MRS of a min-max agent accounts not only for the standard consumption risk kernel, but also for a Knightian uncertainty kernel (Chen and Epstein, 2002, Ulrich, 2012). The Knightian uncertainty kernel captures the time-varying confidence in the accuracy of the most trusted $Q^0$ forecast for future Economic Activities and future Monetary Policy Activities. It is an intuitive learning point that the endogenous innovations in the uncertainty kernel coincide with innovations in Economic Activity and innovations in Monetary Policy Activity. Stochastic volatility in the accumulated amount of model misspecification doubts leads to a heteroscedastic MRS. Notice that this result holds independent of whether or not consumption itself is heteroscedastic.

It is worth emphasizing that exposure to Knightian uncertainty about Economic Activity and about Monetary Policy Activity endogenizes two stochastic market prices of Knightian uncertainty. The dynamic of the MRS reveals this property immediately, i.e.

$$\frac{-d m_t}{m_t} = \rho dt + \frac{dc_t}{c_t} - \frac{d (LR_{t\rightarrow t+\Delta}^r \cdot LR_{t\rightarrow t+\Delta}^{MP})}{LR_{t\rightarrow t+\Delta}^r \cdot LR_{t\rightarrow t+\Delta}^{MP}} = \rho dt + \frac{dc_t}{c_t} + \sqrt{2} \cdot \text{Uncert}_t^r \cdot dW_t^r - \sqrt{2} \cdot \text{Uncert}_t^{MP} \cdot dW_t^{MP}. \quad (27)$$

The market price of consumption risk is constant and positive, i.e. $\sigma_c > 0$. On the contrary, the market price for exposure to Knightian uncertainty about Economic Activity is positive and time-varying, i.e. $\sqrt{2} \cdot \text{Uncert}_t^r \in \mathcal{R}_+$. Equity would pay off a positive equity risk premium because of its positive exposure to Economic Activity (Bansal and Yaron, 2004). On the other hand, Knightian uncertainty about future Monetary Policy Activities pays out a negative but also time-varying market price of uncertainty, $-\sqrt{2} \cdot \text{Uncert}_t^{MP} \in \mathcal{R}_-$. An asset whose return has negative exposure to Monetary Policy Activity shocks would pay a positive equilibrium premium. In Ulrich (2012) this makes nominal Treasury bonds pay out a positive premium, whereas in Ang and Ulrich (2012) this would make equity pay out a positive equity risk premium.
3.5 Endogenous Stochastic Volatility in Marginal Rate of Substitution

We emphasize the empirical puzzle from Section 2, namely that from a variety of traditional macro risk factors it has been only \textit{Uncertainty} which explained a considerable fraction of variations in the volatility of bonds and bond options. In this section we analyze whether our simple general equilibrium model is able to endogenize the same insight.

To investigate the volatility implications further, we want to understand how \textit{Uncertainty} affects the conditional variance of the agent’s \textit{MRS}. The previous equation implies that the conditional variance of the logarithmic \textit{MRS} is affine in the time-varying amount of \textit{Monetary Policy Uncertainty} and \textit{Economic Uncertainty}, i.e.

\[ \text{Var}_t \left( d \ln m_t \right) = \sigma^2_c dt + 2 \cdot (\text{Uncert}_t^r + \text{Uncert}_t^{MP}) \right) dt. \] (28)

The effect of economic and monetary policy uncertainty on the variance of the agent’s \textit{MRS} is clearly visible from the last equation. Two observations are worth mentioning. First, an exogenous rise in either \textit{Economic Uncertainty} or \textit{Monetary Policy Uncertainty} leads to an endogenous increase in the conditional variance of the agent’s \textit{MRS}. A one percent increase in either of these uncertainties leads to an economically meaningful 2% increase in $\text{Var}_t(d \ln m_t)$.

Variations in \textit{Uncertainty} could therefore indeed be potentially very important for understanding variations in the volatility of financial markets.

Second, the equilibrium outcome for $\text{Var}_t(d \ln m_t)$ emphasizes that heteroscedasticity in the agent’s \textit{MRS} might arise from variations in the agent’s confidence in the accuracy of his most trusted workhorse DGP. These fluctuations might have nothing to do with fluctuations in standard measures of fundamental macro \textit{Risk} or fundamental macro \textit{Activity}. An observer, who stands outside of the model (ignoring Knightian uncertainty) would be tempted to term this excess volatility. To be more precise, the economic policy uncertainty factors that drive in our model the volatility in the financial market are of course related to the underlying macro economy, but variations in these factors ($\eta^r$ and $\eta^{MP}$) are not captured by standard fundamental risk factors, such as business cycle risk or macroeconomic volatility risk. Instead, a model must be extended to e.g. Knightian uncertainty, to account for this feature of the data. Patton and Timmermann (2010) conclude in their extended empirical analysis of macroeconomic dispersion data that Knightian uncertainty could indeed be the economic reason for time variation in $\eta^r$ and $\eta^{MP}$. Our economy takes this insight as an exogenous input and shows that the endogenous implications for the volatility of the agent’s \textit{MRS} could be potentially substantial.
3.6 Endogenous Bond Market Volatility

The previous section showed that economic policy Uncertainty drives variations in the conditional variance of the agent’s MRS. This has the intuitive equilibrium implication that the volatility of all financial assets loads on Uncertainty. To further illustrate this observation, we will focus on the volatility of Treasury bonds and bond options and leave it to future research to extend the model to equity options.

Let \( N_t(\tau) \) denote the equilibrium price at time \( t \) of a \( \tau \) maturity zero-coupon Treasury bond. It is well known that the price of such a bond satisfies the following Euler equation

\[
N_t(\tau) = \mathbb{E}_t \left[ m_{t,t+\tau} \frac{p_t}{p_{t+\tau}} \right].
\] (29)

We verify in the appendix that \( N_t(\tau) \) is exponentially affine in four macroeconomic risk factors, namely, (i) Economic Activity, (ii) Monetary Policy Activity, (iii) Economic Uncertainty, and (iv) Monetary Policy Uncertainty. The activity factors enter the price of Treasury bonds because they affect the agent’s inference about future economic growth and about future inflation. Both uncertainty factors enter as endogenous uncertainty premiums into the equilibrium price of bonds, i.e.

\[
N_t(\tau) = e^{A(\tau) + B_\tau(z_t + B_w(\tau) \cdot w_t - B_{hr}(\tau) \cdot \sqrt{2 \cdot \text{Uncert}_t^r} + B_{hMP}(\tau) \cdot \sqrt{2 \cdot \text{Uncert}_t^{MP}}},
\] (30)

where the loadings \( A(\tau) \) and \( B(\tau) \) are deterministic functions of the underlying economy and fully characterized in the appendix.\(^{12}\)

In order to prepare for the upcoming maximum likelihood estimation of the general equilibrium model we solve explicitly for the continuously compounded yield of zero-coupon Treasury bonds. We denote these yields as \( y_t(\tau) = -\frac{1}{\tau} \ln N_t(\tau) \), i.e.

\[
y_t(\tau) = a(\tau) + b_\tau(z_t + b_w(\tau) \cdot w_t - b_{hr}(\tau) \cdot \sqrt{2 \cdot \text{Uncert}_t^r} + b_{hMP}(\tau) \cdot \sqrt{2 \cdot \text{Uncert}_t^{MP}},
\] (31)

where \( a(\tau) = -\frac{A(\tau)}{\tau} \) and \( b(\tau) = -\frac{B(\tau)}{\tau} \).

Notice that the type of Uncertainty matters. Suppose there is an exogenous increase in Economic Uncertainty. This leads to an endogenous increase in the market price of Economic Uncertainty, because the min-max agent worries that future Economic Activity might be systematically lower than previously expected. Investing into Treasury bonds becomes a more...
valuable investment strategy because being long a Treasury bond hedges bad realizations of Economic Activity. A min-max agent hedges the rise in Economic Uncertainty through an increased demand for Treasury bonds. The price of these bonds increases because the negative Economic Uncertainty premium in Treasury bonds increases in absolute magnitude to keep Treasury bonds in zero net supply. The increase in the price of a Treasury bond is consistent with a flight to safety and leads to a steepening of the Treasury yield curve in times of elevated Economic Uncertainty. The yield curve steepens because the impact of the Economic Uncertainty shock is expected to mean revert back to zero, which implies a relatively strong reduction of short-term Treasury yields and a minor reduction in long-term yields.

On the other hand, suppose there is an exogenous increase in Monetary Policy Uncertainty. The negative market price of Monetary Policy Uncertainty increases in magnitude (Proposition 1), because the min-max agent worries to systematically under-predict future inflation. The monetary uncertainties associated with investing into Treasury bonds increase because the real payoff of these bonds is worse than expected in periods where realized trend inflation is higher than predicted. Since Treasury bonds do not hedge the amplified amount of Monetary Policy Uncertainty the price of these bonds drops because the positive Monetary Policy Uncertainty premium in Treasury bonds increases to keep the Treasury bonds in zero net supply. The resulting Monetary Policy Uncertainty premium increases all Treasury yields and leads to an upward sloping yield curve if innovations to Monetary Policy Activities are sufficiently persistent.

Knowing how economic and monetary policy measures for Activity and Uncertainty drive the price of Treasury bonds, allows us to determine the conditional variance of bond returns (quadratic variation in instantaneous bond returns). We denote this instantaneous return variance of a \( \tau \) maturity Treasury bond as \( RV ar_t(N_t(\tau)) \) to emphasize its close relation to the realized volatility measure in Section 2, i.e.

\[
RV ar_t(N_t(\tau)) = (B_w(\tau) \cdot \sigma_w + B_z(\tau) \cdot \sigma_{z2})^2 + B^2_z(\tau) \cdot \sigma_{iz}^2
+ B^2_{h_r}(\tau) \cdot \sigma^2_{gr} \cdot (m_r)^2 \cdot \eta_t^r + B^2_{h_M}(\tau) \cdot \sigma^2_{MMP} \cdot (m_{MMP})^2 \cdot \eta_{MMP}^t, \tag{32}
\]

where the \( B(\tau) \) loadings coincide with the corresponding loadings from the price of the bond.

Consistent with empirical evidence in Section 2, we see that only aggregate Uncertainty drives the realized variance of Treasury bond returns. A positive shock to either type of Uncertainty leads to an endogenous increase in the variance of bond returns. This happens because an increase in Uncertainty increases not only the discount rates (market price of uncertainty) but it also increases the conditional variance of the agent’s MRS, which increases the conditional variance of bond returns.
We now want to use the equilibrium insights to better understand the empirical observation from Section 2. We uncovered that most of the variations in option implied bond volatilities arise from variations in the realized volatility of bond returns. A more skill-full expression for this observation is that we want to understand why bond volatilities move similarly under the physical and under the pricing probability measure. We deliberately abstract from a variance risk premium (see discussion in Section 2) which as a consequence generates the high covariation between the physical and the pricing dynamic of volatility. An analytically convenient consequence of this modeling choice is that the option implied variance on a bond coincides with the expected variance of the bond return (under the physical probability \( Q^0 \)), i.e.

\[
E_t [RVaR_{t+m}(N_{t+m}(\tau))] = (B_w(\tau) \cdot \sigma_w + B_z(\tau) \cdot \sigma_{2z})^2 + \sum_{i \in \{r, MP\}} B_{h_i}(\tau) \cdot \sigma_{\eta_i}^2 \cdot m_i \cdot \left[ \left( m_i \cdot \frac{a_{\eta_i} \cdot m_i}{\kappa_{\eta_i}} \right) \cdot e^{\kappa_{\eta_i} \cdot m_i} - \frac{a_{\eta_i} \cdot m_i}{\kappa_{\eta_i}} \right].
\]

The m-period ahead variance forecast of a \( \tau \) period Treasury bond coincides in the model with the m-period option implied variance of a \( \tau \) maturity Treasury bond. The option implied bond volatility is time-varying and increases in periods where economic and/or monetary policy uncertainty rises. The model explains naturally why an elevated amount of economic and monetary policy Uncertainty leads to an increase in the price of bond options. The persistence of both types of Uncertainty, i.e. \( \kappa_{\eta_r} \) and \( \kappa_{\eta_{MP}} \) and the scaling parameter \( m^r \) and \( m^{MP} \), determine how strongly both uncertainties affect bond options. The persistence of both uncertainties is pinned down through the time-series of \( \eta_r \) and \( \eta_{MP} \), whereas the magnitude of both scaling parameters must be identified through a structural model estimation.

4 Structural Estimation of General Equilibrium Model

The previous section showed that Knightian uncertainty about Monetary Policy Activity and Economic Activity is indeed able to endogenize the empirical bond volatility pattern that we document in Section 2 of this paper. As a last insight we would like to use the equilibrium model to understand whether Economic Uncertainty and Monetary Policy Uncertainty affect yields and volatilities the same way or whether there are meaningful differences. To do so, we estimate the model with the following data, which are laid out in more detail in the appendix.

First, we use continuously compounded yields on Treasury zero-coupon bonds with maturities of 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 years. The data is taken from the Gurkaynak et al. (2007)
database and comprises the years 1972:Q1 to 2009:Q2. Second, we use all four bond volatility measures that were introduced in Section 2. In short, these were realized quarterly volatility of the 4.5-year and the 6.5-year zero-coupon Treasury bond, as well as the option implied volatility on the 5-year and on the 10-year Treasury note. The option implied volatility coincides with the TIV series in Bloomberg. All four volatility time series start in 1994:Q1, which is the earliest date for which we have access to all four volatility measures.

Third, we use two macroeconomic measurement equations to identify \( d \ln c_t \) and \( d \ln p_t \). We match the former with realized quarterly GDP growth and the latter with the GDP implicit inflation rate. The FRED database from the St. Louis Federal Reserve publishes both time series. Fourth, our four, in real-time observable, state equations coincide with the Activity and Uncertainty measures from Section 2. In summary, these are (i) Economic Activity \((z_t)\), (ii) Monetary Policy Activity \((w_t)\), (iii) Economic Uncertainty \((\eta^r_t)\), and (iv) Monetary Policy Uncertainty \((\eta^MP_t)\). This data is from 1972:Q1 to 2009:Q2. The estimation starts in 1972:Q1 because of the real-time survey data that we rely upon to construct the four state equations.

We use maximum likelihood to estimate the model. Four observable and conditionally Gaussian state variables are employed to fit 16 measurement equations. We summarize the parameters of the economy in a vector \( \theta \) and all four state equations in a vector \( s_t \). The appendix verifies that the joint logarithmic likelihood function, \( \ln L(\theta) \), coincides with

\[
\ln L(\theta) = \sum_{t=1}^{T-1} \ln f_m(m_{t+1}|\{z_t, w_t; \theta\}) + \ln f_s(s_{t+1}|\{s_t; \theta\}) \\
+ \ln f_{u^y}(u^y_{t, \tau}|\{s_t; \theta\}) + \ln f_{u^{RVol}}(u^{RVol}_{t, \tau}|\{\eta^MP_t, \eta^r_t; \theta\}) + \ln f_{u^{TIV}}(u^{TIV}_{t, \tau}|\{\eta^MP_t, \eta^r_t; \theta\})
\]

(34)

where all the marginal transition densities, \( f_m, f_s, f_{u^y}, f_{u^{RVol}}, f_{u^{TIV}} \) are fully characterized in the appendix. The marginal density \( f_m \) captures the joint likelihood of \( d \ln c_t \) and \( d \ln p_t \), \( f_s \) captures the joint likelihood of the four state processes, \( f_{u^y} \) stands for the joint likelihood of the ten Treasury yields, \( f_{u^{RVol}} \) stands for the joint likelihood of the two realized bond volatilities, and last but not least, \( f_{u^{TIV}} \) captures the joint likelihood of the two option implied bond volatilities.

The empirical fit of the model is surprisingly good, considering that (i) it only uses four observable state processes, (ii) it matches Treasury yields and Treasury volatilities, and (iii) it matches option implied bond volatilities with a zero variance risk premium. How does the model achieve that? The empirical insights of Section 2 helped to identify economic and monetary policy Activity and Uncertainty as the key macro risk factors. Moreover, Section 2 identified that the dynamic of the sample variance risk premium is negligible.
4.1 Cross-Section of Bond Volatility

Figure 5 plots the estimated model implied term structure of bond volatilities and option volatilities together with the data counterpart and standard error bounds. The upper left panel shows that the average fit to the physical bond volatility is very accurate. The annualized physical volatility of the 4.5-year zero-coupon bond is roughly 4%, whereas it is roughly 6% for the 6.5-year zero-coupon bond. Notice that we explained in Section 2 that we do not compare the 5 (10)-year TIV with the physical volatility of the 5 (10)-year zero-coupon Treasury volatility, but that instead we match their effective duration, as laid out in the Appendix. The upper right panel shows that the model is indeed able to explain the term structure of option implied volatilities with only Uncertainty and with a zero variance risk premium. The lower left and the lower right panel plot the model implied term structure for physical bond volatilities and option implied volatilities, for the period 1972:Q1 to 2009:Q2. Both term structures are strongly upward sloping with a similar shape to the 1994 to 2009 data.

Figure 6 shows that the fit to the yield curve is successful. The model implied yield curve is clearly upward sloping and all yields are within standard confidence error bounds. The fit to the 4.5 and 6.5-year yield is actually very good, which is comforting because we also match the 4.5 and the 6.5-year interest rate volatility and the corresponding option implied volatility.

The parameter estimates in Table 7 reveal that the yield curve model is effectively a two-factor model, whereas the volatility model is effectively a one-factor model. The very persistent Monetary Policy Activity measure, is known to be the most important observable macroeconomic yield curve factor (Ulrich, 2012, Ang and Ulrich, 2012). Our estimation confirms that and stresses that Monetary Policy Uncertainty is an important volatility factor.

It is interesting to note that if one considers Uncertainty to be an aggregate macro factor with two subcomponents, our estimation argues that Economic Uncertainty is the high frequency uncertainty component, whereas Monetary Policy Uncertainty is the lower frequency component. The half-life of a shock to Monetary Policy Uncertainty is over 50 years. This is a potentially alarming finding for policy makers because it indicates that increases in Monetary Policy Uncertainty are extremely long lasting.

4.2 Bond Volatility and Monetary Policy Uncertainty

We next turn to the question whether the model provides any insights to why volatility, as documented in Section 2, appears to be underpriced in 2004 to 2007. Figure 7 plots physical volatil-
ity (top panel), option implied volatility (middle panel), and the corresponding Treasury yield (lower panel) for the 4.5-year Macaulay duration bond (left panels) and the 6.5-year Macaulay duration bond (right panels). The physical bond volatility moves very closely to the priced volatility, for all maturities, as already noted in Section 2.

Three volatility related observations are worth mentioning. First, physical and priced volatility increase sharply right before recessions. The model captures this stylized fact accurately. That is a striking observation because (i) the model implied volatilities are mainly driven by Monetary Policy Uncertainty, and (ii) it emphasizes that spikes in option implied bond volatility seems to arise from spikes in physical volatility.

Second, a strong divergence in model implied and data observed volatility starts in 2004 and lasts until the middle of 2007. With the help of the model we can infer that the underestimation of physical bond volatility was then transferred into lower option volatilities. Both volatility markets were affected. So it was not that the market price of volatility risk was too low, instead the estimated amount of volatility risk was too low, compared to the amount of Monetary Policy Uncertainty in the U.S. economy. The under-prediction of volatility was sharply reversed in late 2007, when the financial crisis started to arrive. The sudden increase in fear is apparent because physical and priced volatility started to overshoot the model counterpart.

Third, for the moment it remains a puzzle why physical volatility started to drop significantly in 2004, compared to the observed amount of Monetary Policy Uncertainty. Even more puzzling is the observation that the ‘artificially low’ Treasury yields (according to our model) after the 2001 recession, started to revert back to fundamentals (our model) in 2004. There is a striking negative correlation between the yields coming back to fundamentals and the volatility deviating from fundamentals. In other words, the Federal Reserve started to raise interest rates in 2004 to correct the path of easy monetary policy after 2001, which the volatility market interpreted as a signal towards lower financial market risk ahead. But according to our model, the amount of Monetary Policy Uncertainty remained unchanged which should have led volatility in the bond market to remain unchanged. The market noticed the underpricing of risk when the financial crisis hit unexpectedly and when monetary policy prepared for the launch of untested Quantitative Easing policies.

When the financial crisis hit in late 2007, the Federal Reserve went back to the policy of easy money, which in our estimation is clearly visible, because even long-term rates started to be again substantially lower than what our model with Economic Activity and Monetary Policy Activity justifies. At the same time, volatility in the market starts to overshoot compared
to the amount of Monetary Policy Uncertainty in the economy. This indicates a complex interaction between actions of the Federal Reserve and the volatility of bond markets which our consumption-based asset pricing model captures partly. We leave it to future research to further analyze this identified puzzle.

Figure 8 shows that the term structure of volatility in the bond and in the option market was noticeably higher during the Monetary Policy Experimentation than during the Great Moderation. The model has been estimated for the entire sample and sample averages are used to find the term structure for these subperiods. The longer the duration of the underlying Treasury bond, the higher the volatility spread. The average bond volatility for a 10-year Macaulay duration bond has been 10.72% during the Monetary Policy Experimentation and 8.35% during the Great Moderation. The corresponding average option implied volatility has been 11.09% during the Monetary Policy Experimentation and 8.83% during the Great Moderation. Notice that these volatility spreads arise because of variations in the observed amount of Monetary Policy Uncertainty.

A variance decomposition of the model implied bond and option implied volatility reveals that the main uncertainty driver of bond volatility is Monetary Policy Uncertainty. The intuition for this model implication is very instructive. Fear about under-predicting future Monetary Policy Activity leads to an upward sloping nominal yield curve (Ulrich, 2012), whereas we find in this paper that fear about over-predicting future Economic Activity leads to a downward sloping nominal yield curve. It is a stylized empirical fact that the U.S. Treasury yield curve is on average upward sloping. Monetary Policy Uncertainty endogenizes this stylized fact and makes the bond volatility be time-varying. Economic Uncertainty can endogenize time-varying volatility as well, but at the cost of a downward sloping Treasury yield curve. We match in our structural model estimation both, the dynamics of the Treasury yield curve and the dynamics of the volatility in bond and option markets. In retrospect it is therefore not a surprise that Monetary Policy Uncertainty dominates Economic Uncertainty.

The dominance of Monetary Policy Uncertainty for the volatility of bond options is depicted in Figure 9. It plots the time series for the model implied volatility on a one month option written on a 10-year Treasury note together with our observable measure of Monetary Policy Uncertainty. The unconditional correlation of 99% confirms that Monetary Policy Uncertainty is the main driver in the model for the volatility of bond and bond option markets. Periods of spikes in Monetary Policy Uncertainty, such as the 1981, 1991, and 2008 recessions are periods where the volatility of options spike as well.
We plot in Figure 10 several financial market responses to a one standard deviation shock to $\eta_{t}^{MP}$. This figure reveals that shocks to Monetary Policy Uncertainty have a severe effect on the financial market. The slope of the yield curve would increase by 0.2% as the monetary policy ambiguity premium steepens (Ulrich, 2012, Wright 2011). The steepening of the yield curve is accompanied by a 4.74% increase in the physical volatility of a 4.5-year Macaulay duration bond, and a 5.99% increase in the physical volatility of a 6.5-year Macaulay duration bond. Option implied volatilities on bonds with the same Macaulay duration would increase by the same amount because we abstract from a variance risk premium (see Section 2).

The estimated high persistence of $\eta_{t}^{MP}$ makes the impact on the slope and the volatility of interest rates to be long lasting.

### 4.3 Detection Error Probability

One natural question of interest remains: how much Knightian uncertainty has been necessary to endogenize the above mentioned equilibrium results? The appendix derives the analytical expression for the detection error probability (DEP). Detection error probabilities are an intuitive tool, because they quantify the probability that an econometrician could, after seeing the entire sample of the state vector, tell whether $Q^0$ or $Q^h$ generated the data. Notice that two Gaussian transition densities are indistinguishable if their DEP is 50%. According to Hansen and Sargent (2008) a DEP of 10% or more poses a realistic threat of choosing the wrong state vector DGP.

Our maximum likelihood estimation implies a DEP of 21.65%. This says that there is a 21.65% chance that an econometrician chooses the wrong DGP for Economic Activity and Monetary Policy Activity, after seeing the entire state vector. This implies that the agent has been confronted with a lot of Knightian uncertainty about whether or not $Q^0$ or $Q^h$ generated the observed economic and policy data.

### 5 Conclusion

Our analysis started out documenting that roughly half of the quarterly variations in option implied bond volatilities and interest rate volatilities correlate with real-time measures of economic and monetary policy uncertainty. Standard measures of macroeconomic volatility and business cycle risk did not contribute to explain variations in those bond volatilities.

---

13 The result would also hold in a model with a non-zero variance risk premium that is driven by a factor different than Monetary Policy Uncertainty.
We propose a simple and extremely parsimonious general equilibrium asset pricing model that is able to explain why economic and monetary policy uncertainty matter so strongly for the volatility of bond and option markets. Estimating the equilibrium model with maximum likelihood reveals that Monetary Policy Uncertainty is the most important macroeconomic volatility risk factor in bond markets. A one standard deviation shock to Monetary Policy Uncertainty increases the slope of the yield curve by 0.2% and raises the (physical) volatility and option implied (priced) volatility of a 6.5-year Macaulay duration bond by an economically meaningful 6%.

We conclude first, that volatility in bond markets suggests that Knightian uncertainty is far more important that traditional measures of volatility risk and expected business cycle risk. Second, a simple general equilibrium asset pricing model with only observable risk factors can account for the key dynamics in the cross-section and time-series of Treasury interest rates, interest rate volatility, and option implied volatility. We leave an extension towards equity options and towards analyzing the variance risk premium to future research.
6 Appendix

"For Online Publication"

6.1 Macaulay Duration for Matching Interest Rate Risk

The underlying contract of a $TIV(5 - yr) (TIV(10 - yr))$ is a 6% Treasury coupon bond with not more than 5 years and 3 months (ten years) and at least 4 years and 2 months (6.5 years) time to maturity. We use the following approximation to match the TIV implied volatility to the volatility of an option on a zero-coupon bond.

First, the coupon rate of the underlying is fixed to be 6%. Second, as the maturity of the deliverable claim, we use the average of 4 and 5 years for the TIV(5-yr) (i.e. 4.5) and the average of 6 and 10 years for the TIV(10-yr) (i.e. 8). Third, the time span of the option data is 1994:Q1 to 2009:Q2. The average 8 year (4.5) Treasury yield has been 5% (4.6%) during this time period. Fourth, the Macaulay duration of a Treasury note with 6% coupon, maturity in 8 years, which trades at a yield of 5% is approximately 6.5. Analogously, the Macaulay duration of a Treasury note with 6% coupon, maturity in 4.5 years, which trades at a yield of 4.6% is approximately 4.5.

To summarize, the underlying of a TIV(5-yr) and TIV(10-yr) contract has had approximately the same interest rate risk in 1994:Q1 to 2009:Q2 than a 4.5-yr and 6.5-yr Treasury zero-coupon bond, respectively.

6.2 Data

State Variables

We use real-time forecast data from the Survey of Professional Forecasters to identify the empirical counterpart of $w_t, z_t, \eta_{t}^{MP}, \eta_{t}^{r}$. We match $w_t (z_t)$ with the demeaned median one quarter ahead inflation (GDP growth) forecast and we match $\eta_{t}^{MP} (\eta_{t}^{r})$ with the cross-sectional dispersion of one quarter ahead inflation (GDP growth) forecasts. We remove seasonality in the cross-sectional dispersion data by using a 4-quarter moving average. All data is from first quarter 1972 to second quarter 2009.

Measurement Variables: Treasury Bond Yields

Data for continuously compounded Treasury yields on nominal zero-coupon bonds is taken from the Board of Governors of the Federal Reserve System and is based on work in Gurkaynak, Sack, and Wright (2007). We use yields of maturities 1,2,3,4,5,6,7,8,9,10 years. Data is from first quarter 1972 to second quarter 2009. provide details

Measurement Variables: Realized Bond Volatility and Option Implied Bond Volatility

31
We use the above mentioned data on Treasury bond yields (with daily frequency) to construct daily prices of zero-coupon bonds. We approximate for the quarterly variance of Treasury zero-coupon bonds by taking the average squared logarithmic bond return within one quarter.

Option implied bond volatility is taken from the TIV data series that is provided by Bloomberg. TIV is the bond market analog to the CBOE VIX measure. The TIV measures the implied volatility of a one month option on a Treasury Note futures. We use $TIV(5-yr)$ and $TIV(10-yr)$. The underlying contract of a $TIV(5-yr)$ ($TIV(10-yr)$) is a coupon Treasury bond with not more than 5 years and 3 months (ten years) and at least 4 years and 2 months (6.5 years) time to maturity.

We explained above that the Macaulay duration of the $TIV(5-yr)$ ($TIV(10-yr)$) series in my sample is $4.5$ ($6.5$), respectively. Treasury bond data from Gurkaynak, Sack, and Wright (2007) represent prices of zero-coupon bonds. For consistency reasons, we treat the $TIV(5-yr)$ ($TIV(10-yr)$) data to coincide with the option implied volatility on a $4.5$ ($6.5$) Macaulay duration bond.

This means we use 4 measurement equations. We use the quarterly realized bond volatility for zero-coupon bonds with a Macaulay duration (maturity) of $4.5$ and $6.5$ years. We also use the $TIV(5-yr)$ and $TIV(10-yr)$ series.

### 6.3 Proof of Proposition 1

Rewrite the constrained minimization in (19) as a relative entropy constrained HJB. $J$ denotes the value function. It depends on $J = J(\ln c_t, \eta^t_{MP}, \eta^r_t, z_t)$. The time varying Lagrange multipliers for the entropy constraints are $\theta^r_t$ and $\theta_{MP}^t$.

$$pJ(\ln c_t, \eta^t_{MP}, \eta^r_t, z_t) = \min_{\theta^r_t, \theta_{MP}^t} \theta^r_t \left( \frac{(h^r_t)^2}{2} - \frac{(m^r)^2}{2} (\eta^r_t)^2 \right) + \theta_{MP}^t \left( \frac{(h^t_{MP})^2}{2} - \frac{(m_{MP})^2}{2} (\eta^t_{MP})^2 \right) + \mathcal{A}^h J(\ln c_t, \eta^t_{MP}, \eta^r_t, z_t),$$

(35)

where $\mathcal{A}^h$ is the second order differential operator (under the ambiguity adjusted measure) applied to the value function $J$. Guess the value function is linear in the states, i.e. $J = \delta_0 + \delta_z z_t + \delta_{MP} \eta^t_{MP} + \delta_{Ir} \eta^r_t$. The second order differential operator applied to the value function is

$$\mathcal{A}^h J = \delta_z (\kappa_z + \sigma_{1z} h^r_t + \sigma_{2z} h^t_{MP}) + \delta_{Ir} (a_{Ir} + \kappa_{Ir} \eta^r_t) + \delta_{MP} (a_{MP} + \kappa_{MP} \eta^t_{MP})$$

(36)

First-order conditions with regard to $h^r_t$ and $\theta^r_t$ reveal

$$\theta^r_t = \frac{-\sigma_{1z} \delta_z}{\pm m^r \eta^r_t} \quad (37)$$

$$h^r_t = \pm m^r \eta^r_t \quad (38)$$
Note \((\delta_z > 0, \sigma_{1z} > 0)\), the robust HJB is minimized at

\[
h_t^r = -m^r \eta_t^r, \quad m^r \in \mathcal{R}^+
\]

\[
\theta_t^r = -\frac{\sigma_{1z}\delta_z}{m^r \eta_t^r} = \frac{b_0}{\eta_t^r}, \quad b_0 \in \mathcal{R}^+,
\]

where we defined \(b_0 \equiv -\frac{\sigma_{1z}\delta_z}{m^r} > 0\). This proves the first part of the proposition. 

First-order conditions with regard to \(h_t^{MP}\) and \(\theta_t^{MP}\) reveal

\[
\theta_t^{MP} = -\frac{-\sigma_{2z}\delta_z}{m^{MP} \eta_t^{MP}} \quad \text{and} \quad h_t^{MP} = \pm m^{MP} \eta_t^{MP}.
\]

Note \((\delta_z > 0, \sigma_{2z} < 0)\), the robust HJB is minimized at

\[
h_t^{MP} = m^{MP} \eta_t^{MP}, \quad m^{MP} \in \mathcal{R}^+
\]

\[
\theta_t^{MP} = -\frac{\sigma_{2z}\delta_z}{m^{MP} \eta_t^{MP}} = \frac{b_1}{\eta_t^{MP}}, \quad b_1 \in \mathcal{R}^+,
\]

where we defined \(b_1 \equiv -\frac{\sigma_{2z}\delta_z}{m^{MP}} > 0\). This proves the remaining part of the proposition. Plug the solution to the robust HJB and verify that the guess of the linearized value function was correct.

### 6.4 Derivation of Nominal Bond Yields

Let \(F = F_t(\tau)\) be the price of a nominal bond. In the economy this price is exponentially affine \(F = e^{A(\tau) + B(\tau)X}\) where \(X\) denotes the state vector \(X_t = (w_t, z_t, h_t^{MP}, h_t^r)'\). Note that \(h_t^i \forall i \{r, MP\}\) solves

\[
dh_t^i = (a_{\eta_t^i} m_t^i + \kappa_{\eta_t^i} h_t^i)dt + \sigma_{\eta_t^i} \sqrt{m_t^i h_t^i} dW_t^{\eta_t^i}, \forall i \in \{r, MP\},
\]

F solves the following PDE

\[
R : F = A^H F + F_t, \quad F_t = -F_{\tau}
\]

where \(R\) is the nominal short rate, i.e. \(R_t dt = -E_t \left[ \frac{d(m_t/p_t)}{m_t/p_t} \right], A^H\) is the second order differential operator and \(F_{\tau}\) is the first derivative of \(F\) with regard to \(\tau\). Using the equilibrium nominal short rate and the exogenous dynamics of \(X\) reveals that the bond loadings solve simple ordinary differential equations. The solution to the loadings \(B_w(\tau)\) and \(B_z(\tau)\) is:

\[
B_w(\tau) = \frac{1}{\kappa_w} (1 - e^{\kappa_w \tau})
\]

\[
B_z(\tau) = \frac{1}{\kappa_z} (1 - e^{\kappa_z \tau}).
\]
The loading $B_h(\tau)$ solves a one-dimensional Riccati equation. We obtain an analytical solution by approximating $B_z(\tau)$ with its steady-state value $B_z(\infty)$:

$$B_h(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \quad (49)$$

$$g := -\beta_1 + d$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \quad (50)$$

$$\beta_0 := \frac{\sigma_z}{\kappa_z} \quad (52)$$

$$\beta_1 := \kappa_\eta$$

$$\beta_2 := \frac{1}{2}m^r\sigma_\eta^2. \quad (53)$$

The loading $B_{MP}(\tau)$ solves a one-dimensional Riccati equation. We obtain an analytical solution by approximating $B_w(\tau)$ with its steady-state value $B_w(\infty)$:

$$B_h(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \quad (55)$$

$$g := -\beta_1 + d$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \quad (56)$$

$$\beta_0 := \frac{\sigma_z}{\kappa_z} + \frac{\sigma_w}{\kappa_w} \quad (58)$$

$$\beta_1 := \kappa_\eta$$

$$\beta_2 := \frac{1}{2}m^{MP}\sigma_\eta^{MP}. \quad (59)$$

Function $A(\tau)$ follows from direct integration.

### 6.5 Derivation of Detection Error Probability

The derivation of the detection-error probabilities $p_T(m^r, m^{MP})$ follows directly from Maenhout (2006):

$$p_T(m^r, m^{MP}) = \frac{1}{2} \left( Pr \left( \ln \frac{dQ_h^T}{dQ_T^0} > 0 | dQ_0^0, F_0 \right) + Pr \left( \ln \frac{dQ_h^T}{dQ_T^0} > 0 | dQ^h, F_0 \right) \right) \quad (61)$$

$$= \frac{1}{2} \left( Pr \left( -\frac{1}{2} \int_0^T h'_m h_m dm + \int_0^T h_m \cdot dW^z_m > 0 | dQ_0^0, F_0 \right) \right)$$

$$+ \frac{1}{2} \left( Pr \left( -\frac{1}{2} \int_0^T h'_m h_m dm - \int_0^T h_m \cdot dW^{z,h}_m > 0 | dQ^h, F_0 \right) \right) \quad (62)$$

where $h_t = (m^{MP}\eta^{MP} m^r\eta^r)'$ is the endogenous distortion to trend GDP growth. The last equation coincides with

$$p_T(m^r, m^{MP}) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left( Re \left( \frac{\phi(h,k,0,T)}{ik} \right) - Re \left( \frac{\phi(k,0,T)}{ik} \right) \right) dk \quad (63)$$
In order to get an analytical solution, we approximate the conditional volatility of the uncertainty
pre-
φ
plex valued Riccati equations. We provide details on the derivation of the Riccati equations for
φ
E
F
where
G
i
h
in the amount of inflation distortion
φ
Set
b
φ
τ
φ
h
Application Feynman-Kac theorem to \( \phi^h \) and \( \phi \) reveals that they are an exponentially quadratic function
in the amount of inflation distortion \( h_t \):

\[
\phi^h(k, t, T) = e^{ik + \frac{1}{2} G(\tau, k) + \sum_{j \in \{MP,r\}} E_j(\tau, k) h_j(t) + \frac{1}{2} \sum_{j \in \{MP,r\}} \frac{F_j(\tau, k)}{2} h_j^2(t)}
\]

(64)

\[
\phi(k, t, T) = e^{ik + \frac{1}{2} G(\tau, k) + \sum_{j \in \{MP,r\}} \hat{E}_j(\tau, k) h_j(t) + \frac{1}{2} \sum_{j \in \{MP,r\}} \frac{\hat{F}_j(\tau, k)}{2} h_j^2(t)}
\]

(65)

\[
z_T := e^{\xi_{1,T}},
\]

(66)

where \( G(\tau, k), E_j(\tau, k), F_j(\tau, k), \hat{G}(\tau, k), \hat{E}_j(\tau, k), \hat{F}_j(\tau, k) \) are deterministic solutions to standard complex valued Riccati equations. We provide details on the derivation of the Riccati equations for \( \phi^h \). The derivation of \( \phi \) is analogous.

In order to get an analytical solution, we approximate the conditional volatility of the uncertainty premium by its steady state value, i.e.

\[
dh_t = (a_\eta m + \kappa_\eta h_t) dt + \sigma_\eta \sqrt{\bar{m} h_t} dW^h_t
\]

(67)

\[
\approx (a_\eta m + \kappa_\eta h_t) dt + \sigma_\eta \sqrt{\bar{m} a_\eta / (-\kappa_\eta)} dW^w_t
\]

(68)

For ease of notation we define \( b := \sqrt{\bar{m} a_\eta / (-\kappa_\eta)} \), where more specifically \( b_r \) refers to the conditional steady state volatility of \( dh^r \) and \( b_w \) is the analog for \( dh^w \).

\( \phi^h(k, t, T) \) solves \( \phi^h_t = \mathcal{A} \phi^h \) where \( \tau = T - t \) and \( \phi^h_t \) stands for \( \frac{\partial \phi^h}{\partial t} \).

\[
\phi^h_t = \phi^h \left( \hat{G}(\tau, k) + \sum_{j \in \{MP,r\}} \hat{E}_j(\tau, k) h_j(t) + \frac{1}{2} \sum_{j \in \{MP,r\}} \hat{F}_j(\tau, k) h_j^2(t) \right)
\]

(69)

\[
\frac{\mathcal{A} \phi^h}{\phi^h} = \sum_{j \in \{MP,r\}} \left[ (E_j(\tau, k) + F_j(\tau, k) h_j(t)) \left( a_{\eta j} m_j + \kappa_\eta h_j(t) \right) \right] + 0.5i k (k + 1) \left( h_{3MP}^2(t) + h_{3r}^2(t) \right)
\]

\[
+ \frac{1}{2} \sum_{j \in \{MP,r\}} \left( E_j^2(\tau, k) + F_j^2(\tau, k) h_j^2(t) + 2E_j(\tau, k) F_j(\tau, k) h_j(t) \right) b_j^2
\]

(70)

Set \( \phi^h_\tau = \mathcal{A} \phi^h \) and match coefficients:

\[
F_j(\tau, k) = F^r_j(\tau, k) + F^c_j(\tau, k)
\]

(71)

\[
F^r_j(\tau, k) = k \cdot \tau
\]

(72)

\[
F^c_j(\tau, k) = \frac{(a_j + d_j)(1 - e^{d_j \tau})}{2b_{2j}^2(1 - g_je^{d_j \tau})}
\]

(73)

where \( F^r \) is the real part of \( F \) and \( F^c \) is the complex part and

\[
a_j = -b^*_j; \quad d_j = \sqrt{a_j^2 - 4b^*_{0j}b^*_{2j}}; \quad g_j = \frac{a_j + d_j}{a_j - d_j}; \quad b^*_{0j} = -k^2
\]

(74)

\[
b^*_{1j} = 2\kappa_\eta; \quad b^*_{2j} = b_{2j}^2
\]

(75)
where $j \in \{MP, r\}$. The stable steady state solution of $F$ is

$$F_j(\infty, k) = \frac{b^r_j + d_j}{2b^r_{2j}}. \quad (76)$$

The loadings $E_j(\tau, k), j \in \{MP, r\}$ solve the following ode

$$\frac{d}{d\tau}E_j(\tau, k) = \kappa_{nj}E_j(\tau, k) + m_\tau a_\tau F_j(\tau, k) + E_j(\tau, k)F_j(\tau, k)b^r_j. \quad (77)$$

We obtain an analytical approximation by approximating $F_j(\tau, k)$ around its steady state value $F_j(\infty, k)$.

$$E_j(\tau, k) = -\frac{\hat{a}_j}{b_j}(1 - e^{\hat{b}_j \tau}) \quad (78)$$

$$\hat{a}_j = F_j(\infty, k)m_\tau a_\tau \quad (79)$$

$$\hat{b}_j = F_j(\infty, k)b^r_j + \kappa_{nj}. \quad (80)$$

The loading $G(\tau, k)$ is obtained through straightforward integration

$$G(\tau, k) = \sum_{j \in \{MP, r\}} \left( m_\tau a_\tau \int_0^\tau E_j(u, k)du \right) + \frac{1}{2} \sum_{j \in \{MP, r\}} b^r_j \int_0^\tau E^2_j(u, k)du. \quad (81)$$

The required expression $\phi^h(k, 0, T)$ is therefore

$$\phi^h(k, 0, T) = e^{G(T, k) + \sum_{j \in \{w, r\}} E_j(T, k)h_j(\infty) + \frac{1}{2} \sum_{j \in \{w, r\}} F_j(T, k)h^2_j(\infty)}, \quad (82)$$

where we assumed that $h_j(0)$ started in its steady state $h_j(\infty) = \frac{m_\tau a_\tau}{\kappa_{nj}}$.

### 6.6 Derivation Likelihood Function

The equilibrium model is estimated with ML. The frequency is quarterly. The macroeconomic measurement equations are

$$m_{t+1} := \begin{pmatrix} \ln \frac{c_{t+1}}{c_t} \\ \ln \frac{p^u_{t+1}}{p^u_t} \end{pmatrix} \sim N \left( \mu_m(t), \Sigma_m \Sigma_m' \right), \quad (83)$$

where $\mu_m(t) := \begin{pmatrix} c_0 + z_t \\ p_0 + u_t \end{pmatrix}$ and $\Sigma_m \Sigma_m' := \begin{pmatrix} \sigma^2_c & 0 \\ 0 & \sigma^2_p \end{pmatrix}$.

We summarize the state variables into a vector $s_t$. Its transition density is conditionally Gaussian

$$s_{t+1} := \begin{pmatrix} u_{t+1} \\ z_{t+1} \\ \eta_{\text{MP}}^{t+1} \\ \eta_{\text{r}}^{t+1} \end{pmatrix} \sim N \left( \mu(s_t), \Sigma_{st} \Sigma_{st}' \right), \quad (84)$$
where \( \mu(s_t) := \begin{pmatrix} \thinspace w_t e^{\kappa_s} \\
 0 \\
 \frac{\alpha_m P}{-\kappa_m P} (1 - e^{\kappa_m P}) + \eta_t^{MP} e^{\kappa_m P} \\
 \frac{\alpha_r}{-\kappa_r} (1 - e^{\kappa_r}) + \eta_r e^{\kappa_r} \end{pmatrix} \) and \( \Sigma_{s_t} \Sigma'_{s_t} := \begin{pmatrix}
\sigma_w^2 & \sigma_u \sigma_Z \\
\sigma_u \sigma_Z & \sigma_Z^2 + \sigma_\eta^2 \\
0 & 0 \\
0 & 0 \end{pmatrix} \).

There are ten measurement equations for nominal bond yields. We denote \( u_{t,\tau}^y \) to be the measurement error at time \( t \) of the \( \tau \)-maturity nominal bond yield and \( y_t(\tau) \) to be the observed nominal bond yield and \( \hat{y}_t^m(\tau) \) to be the model implied bond yield from equation (31),

\[
 u_{t,\tau}^y := y_t(\tau) - \hat{y}_t^m(\tau), \quad u_{t,\tau}^y \sim N \left( 0, \sigma_{ME,y,\tau}^2 \right). \tag{85}
\]

There are two measurement equations related to the realized volatility of bond returns. We denote their measurement error as \( u_{t,\tau}^{RVol} \) and the observed realized volatility as \( RVol_t(\tau) \), and the model counterpart as \( RVol_t^m(\tau) \). The latter coincides with the square-root of the realized variance in equation (32),

\[
 u_{t,\tau}^{RVol} := RVol_t(\tau) - RVol_t^m(\tau), \quad u_{t,\tau}^{RVol} \sim N \left( 0, \sigma_{ME,RVol,\tau}^2 \right). \tag{86}
\]

There are two measurement equations related to the option implied volatility of bond returns. We denote their measurement error as \( u_{t,\tau}^{TIV} \) and the observed realization in data as \( TIV_t(\tau) \), and the model counterpart as \( TIV_t^m(\tau) \). The latter coincides with the square-root of the expected variance in equation (33),

\[
 u_{t,\tau}^{TIV} := TIV_t(\tau) - TIV_t^m(\tau), \quad u_{t,\tau}^{TIV} \sim N \left( 0, \sigma_{ME,TIV,\tau}^2 \right). \tag{87}
\]

We denote the Gaussian transition density of \( m_t, s_t, u_{t,\tau}^y, u_{t,\tau}^{RVol} \), and \( u_{t,\tau}^{TIV} \) as \( f_m, f_s, f_{u^y}, f_{u^{RVol}}, f_{u^{TIV}}, \) respectively. The joint log-likelihood \( \ln L(\theta) \), for a given parameter vector \( \theta \), is given by\(^{14} \)

\[
 \ln L(\theta) = \sum_{t=1}^{T-1} \ln f_m(m_{t+1}|\{z_t, w_t; \theta\}) + \ln f_s(s_{t+1}|\{s_t; \theta\}) \\
 + \ln f_{u^y}(u_{t,\tau}^y|\{s_t; \theta\}) + \ln f_{u^{RVol}}(u_{t,\tau}^{RVol}|\{\eta_t^{MP}, \eta_t^r; \theta\}) + \ln f_{u^{TIV}}(u_{t,\tau}^{TIV}|\{\eta_t^{MP}, \eta_t^r; \theta\}) \tag{88}
\]

\(^{14}\)The sample for the volatility related measurement equations is shorter than for the sample for bond yields. We account for that by adjusting \( T \) accordingly.
with the marginal transition densities being equal to:

\[
\ln f_m(m_{t+1}|\{z_t, w_t; \theta\}) = -\ln(2\pi) - \frac{1}{2} \ln \left( \det \left( \Sigma_m \Sigma'_m \right) \right) - \frac{1}{2} (m_{t+1} - \mu_m(t))' \left( \Sigma_m \Sigma'_m \right)^{-1} (m_{t+1} - \mu_m(t))
\]  

(89)

\[
\ln f_s(s_{t+1}|\{s_t; \theta\}) = -2 \ln(2\pi) - \frac{1}{2} \ln \left( \det \left( \Sigma_s \Sigma'_s \right) \right) - \frac{1}{2} (s_{t+1} - \mu(s_t))' \left( \Sigma_s \Sigma'_s \right)^{-1} (s_{t+1} - \mu(s_t))
\]  

(90)

\[
\ln f_{u^y}(u^y_{t,\tau}|\{s_t; \theta\}) = -5 \ln(2\pi) - \frac{1}{2} \ln \left( \sum_{l=1}^{10} \sigma_{ME,y,l}^2 \right) - \frac{1}{2} \sum_{l=1}^{10} \frac{(u^y_{t,l})^2}{\sigma_{ME,y,l}^2}
\]  

(91)

\[
\ln f_{u^{RVol}}(u^{RVol}_{t,\tau}|\{\eta_{t}, \eta^r_{t}; \theta\}) = -2 \ln(2\pi) - \frac{1}{2} \ln \left( \sum_{l=1}^{2} \sigma_{ME,RVol,l}^2 \right) - \frac{1}{2} \sum_{l=1}^{2} \frac{(u^{RVol}_{t,l})^2}{\sigma_{ME,RVol,l}^2}
\]  

(92)

\[
\ln f_{u^{TIV}}(u^{TIV}_{t,\tau}|\{\eta_{t}, \eta^r_{t}; \theta\}) = -2 \ln(2\pi) - \frac{1}{2} \ln \left( \sum_{l=1}^{2} \sigma_{ME,TIV,l}^2 \right) - \frac{1}{2} \sum_{l=1}^{2} \frac{(u^{TIV}_{t,l})^2}{\sigma_{ME,TIV,l}^2}
\]  

(93)
References


Table 1: Volatility and the Macro Economy

<table>
<thead>
<tr>
<th>Principal Component</th>
<th>Volatility Bond Market</th>
<th>Economic &amp; MP Activity</th>
<th>Economic &amp; MP Risk</th>
<th>Economic &amp; MP Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PC1</td>
<td>PC1</td>
<td>PC1</td>
<td>PC1</td>
</tr>
<tr>
<td>Variance Explained</td>
<td>0.9151</td>
<td>0.5450</td>
<td>0.6472</td>
<td>0.8314</td>
</tr>
<tr>
<td>Correlation</td>
<td>1.0000</td>
<td>0.3512</td>
<td>0.3107</td>
<td>0.7076</td>
</tr>
<tr>
<td></td>
<td>0.3512</td>
<td>1.0000</td>
<td>0.4479</td>
<td>0.5740</td>
</tr>
<tr>
<td></td>
<td>0.3107</td>
<td>0.4479</td>
<td>1.0000</td>
<td>0.5611</td>
</tr>
<tr>
<td></td>
<td>0.7076</td>
<td>0.5740</td>
<td>0.5611</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

This table contrasts the summary statistics for the first principal component of the bond volatility panel with the economic and monetary policy activity, risk, and uncertainty panel. The panel of bond volatilities is depicted in Figure 2. The panel of economic and monetary policy activity, risk, and uncertainty is depicted in Figure 1. The activity factors consist of the median SPF one quarter ahead GDP growth and inflation forecasts. The risk factors coincide with volatility factors of the median SPF one quarter ahead GDP growth and inflation forecasts, extracted with a GARCH(1,1). The uncertainty factors coincide with the cross-sectional dispersion of SPF one quarter ahead GDP growth and inflation forecasts.
We regress the first principal component of the panel of option and Treasury bond volatilities onto various risk and uncertainty factors. The factor *Activity* coincides with the first principal component of the median SPF one quarter ahead GDP growth and inflation forecast. The *Risk* factor coincides with the first principal component of a panel that consists of GARCH(1,1) implied volatility factors extracted from the median one quarter ahead SPF GDP growth and inflation forecast. The factor *Uncertainty* coincides with the first principal component of a panel that consists of the cross-sectional dispersion in GDP growth and inflation forecasts. The factors $y_L$, $y_S$, $y_C$ stand for the first, second, and third principal component of a panel of Treasury interest rates. The data is from 1994:Q1 to 2009:Q2. All factors are standardized to have zero mean and unit variance. Robust t-statistics are reported in square brackets.
Table 3: Decomposing Economic Policy Uncertainty

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level, $y_L$</td>
<td>-0.2139</td>
<td>[-0.9218]</td>
</tr>
<tr>
<td>Slope, $y_S$</td>
<td>0.6400</td>
<td>0.5794</td>
</tr>
<tr>
<td>Curvature, $y_C$</td>
<td>0.2372</td>
<td>[1.2227]</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>40.97%</td>
<td>52.10%</td>
</tr>
</tbody>
</table>

We decompose economic policy uncertainty information into information that is embedded in the yield curve and information that appears ‘unspanned’ by the yield curve. We run the following regressions

$$\text{Uncertainty}_t = \beta_S \cdot y_{S,t} + \epsilon_t^{(1)} \quad (\text{I})$$

$$\text{Uncertainty}_t = \beta_L \cdot y_{L,t} + \beta_S \cdot y_{S,t} + \beta_C \cdot y_{C,t} + \epsilon_t^{(2)} \quad (\text{II})$$

where $\text{Uncertainty}$ coincides with the first principal component of a panel that consists of the cross-sectional dispersion in GDP growth and inflation forecasts, $y_L, y_S, y_C$ coincide with the first, second, and third principal component of a panel of Treasury interest rates. We define $\epsilon_t^{(1)}$ to coincide with $\text{Uncertainty}_{\backslash\{y_S\}}$ and $\epsilon_t^{(2)}$ to coincide with $\text{Uncertainty}_{\backslash\{y_L, y_S, y_C\}}$. The factor $\text{Uncertainty}_{\backslash\{y_S\}}$ stands for uncertainty information that is unspanned by the slope of the Treasury yield curve, whereas $\text{Uncertainty}_{\backslash\{y_L, y_S, y_C\}}$ captures uncertainty information that is orthogonal to the level, slope and curvature of the Treasury yield curve. The data is from 1994:Q1 to 2009:Q2. All factors are standardized to have zero mean and unit variance. Robust t-statistics are reported in square brackets.
We regress the first principal component of the panel of option and Treasury bond volatilities onto various risk and uncertainty factors. The factors $y_L$, $y_S$, $y_C$ stand for the first, second, and third principal component of a panel of Treasury interest rates. The factor Uncertainty$_{{\{y_S\}}}$ coincides with the information in Uncertainty that is orthogonal to information in $y_S$. The factor Uncertainty$_{{\{y_L,y_S,y_C\}}}$ coincides with information in Uncertainty that is orthogonal to the joint information in $y_L$, $y_S$, $y_C$. The data is from 1994:Q1 to 2009:Q2. All factors are standardized to have zero mean and unit variance. Robust t-statistics are reported in square brackets.
Table 5: Economic Drivers of the Level Yield Curve Factor

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity</td>
<td>0.6618</td>
<td>[5.4007]</td>
</tr>
<tr>
<td>Risk</td>
<td>-0.1301</td>
<td>[-1.7869]</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>-1.0376</td>
<td>[-9.9736]</td>
</tr>
<tr>
<td>BondVol</td>
<td>0.1075</td>
<td>[1.1373]</td>
</tr>
<tr>
<td>Economic Activity</td>
<td>-0.4587</td>
<td>[-3.3063]</td>
</tr>
<tr>
<td>Monetary Policy Activity</td>
<td>0.3770</td>
<td>[2.7198]</td>
</tr>
<tr>
<td>Economic Risk</td>
<td>-0.0357</td>
<td>[-0.5828]</td>
</tr>
<tr>
<td>Monetary Policy Risk</td>
<td>-0.0171</td>
<td>[-0.1998]</td>
</tr>
<tr>
<td>Economic Uncertainty</td>
<td>-0.3745</td>
<td>[-3.1173]</td>
</tr>
<tr>
<td>Monetary Policy Uncertainty</td>
<td>-0.6401</td>
<td>[-7.3979]</td>
</tr>
</tbody>
</table>

Adjusted $R^2$ 69.78% 69.69%

We regress the level factor of the yield curve onto several macroeconomic risk factors. The factor Activity coincides with the first principal component of the median SPF one quarter ahead GDP growth and inflation forecast. The factor Risk coincides with the first principal component of a panel that consists of GARCH(1,1) implied volatility factors extracted from the median one quarter ahead SPF GDP growth and inflation forecast. The factor Uncertainty coincides with the first principal component of a panel that consists of the cross-sectional dispersion in GDP growth and inflation forecasts. The data is from 1994:Q1 to 2009:Q2 to make it comparable with the bond volatility data. All factors are standardized to have zero mean and unit variance. Robust t-statistics are reported in square brackets.
### Table 6: Economic Drivers of the Variance Risk Premium

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Activity</td>
<td>-0.1138</td>
<td>0.0064</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-1.1125]</td>
<td>[-0.0433]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk</td>
<td>-0.2916</td>
<td>-0.3148</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[1.9327]</td>
<td>[-2.0658]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uncertainty</td>
<td>-0.1367</td>
<td>0.0363</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-0.8819]</td>
<td>[0.28748]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>1.29%</td>
<td>8.50%</td>
<td>1.87%</td>
<td>5.51%</td>
</tr>
</tbody>
</table>

We regress the first principal component of the panel of 5-year and 10-year bond option implied variance risk premiums on several economic risk factors. The factor *Activity* coincides with the first principal component of the median SPF one quarter ahead GDP growth and inflation forecast. The factor *Risk* coincides with the first principal component of a panel that consists of a GARCH(1,1) implied volatility factors extracted from the median one quarter ahead SPF GDP growth and inflation forecast. The factor *Uncertainty* coincides with the first principal component of a panel that consists of the cross-sectional dispersion in GDP growth and inflation forecasts. The data is from 1994:Q1 to 2009:Q2. All factors are standardized to have zero mean and unit variance. Robust t-statistics are reported in square brackets.
Table 7: Parameter Estimates

State Parameters

<table>
<thead>
<tr>
<th>$w$</th>
<th>$z$</th>
<th>$\eta$</th>
<th>$\eta^{MP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>$-0.006107$</td>
<td>$-2.417790$</td>
<td>$-2.446220$</td>
</tr>
<tr>
<td></td>
<td>$(0.000457)$</td>
<td>$(0.082165)$</td>
<td>$(0.017742)$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-$</td>
<td>$-$</td>
<td>$0.004711$</td>
</tr>
<tr>
<td></td>
<td>$(-$)</td>
<td>$(-$)</td>
<td>$(0.006226)$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$0.000824$</td>
<td>$0.000530$</td>
<td>$-0.001865$</td>
</tr>
<tr>
<td></td>
<td>$(0.000006)$</td>
<td>$(0.000003)$</td>
<td>$(0.000029)$</td>
</tr>
</tbody>
</table>

Consumption and Inflation

<table>
<thead>
<tr>
<th>$\ln C$</th>
<th>$\ln p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{const}$</td>
<td>$0.005486$</td>
</tr>
<tr>
<td></td>
<td>$(0.000315)$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$0.005377$</td>
</tr>
<tr>
<td></td>
<td>$(0.004052)$</td>
</tr>
</tbody>
</table>

Economic Policy Uncertainty and $\rho$

<table>
<thead>
<tr>
<th>$m^{MP}$</th>
<th>$m^\tau$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.208350$</td>
<td>$205.548000$</td>
<td>$0.000503$</td>
</tr>
<tr>
<td>$(0.169886)$</td>
<td>$(5.477361)$</td>
<td>$(0.000222)$</td>
</tr>
</tbody>
</table>

We report parameter estimates and asymptotic standard errors. The model is estimated with maximum likelihood. The asymptotic covariance matrix of the estimated parameters coincides with the inverse of the outer product of the gradient of the likelihood function. The standard error is the square-root of the corresponding diagonal element of that matrix. The data is from 1972.Q1 to 2009.Q2.
Figure 1: Economic and Monetary Policy Risk Measures

This figure plots the z-score of different measures of economic and monetary policy risks. The upper panel plots the median one quarter ahead GDP growth and inflation forecast, respectively; published by the Survey of Professional Forecasters. The former measures anticipated economic activity, whereas the second measures anticipated monetary policy activity. The medium panel plots stochastic volatility of expected inflation and expected GDP growth, extracted from a GARCH(1,1) applied to the above mentioned economic and monetary policy activity measures. The lower panel plots the z-score of the cross-sectional dispersion of the above mentioned one quarter ahead GDP growth and inflation forecasts, respectively. The latter measures economic and monetary policy Knightian uncertainty. The sample is from 1994:Q1 to 2009:Q2, the frequency is quarterly, and NBER recession dates are shaded.
This figure plots the zscore of different volatility measures in the bond market. The time-series for the ‘TIV’ data coincides with the option implied volatility on a 5- and 10-year Treasury note. This data is from Bloomberg. The underlying Treasury note of the option contract is a hypothetical 6% coupon bond with par value of $100. I determine the Macaulay Duration of these instruments and plot the realized volatility of zero-coupon bonds who share the same Macaulay Duration. The zero-coupon bond data is from the Federal Reserve database. The sample is from 1994:Q1 to 2009:Q2, the frequency is quarterly, and NBER recession dates are shaded.
Figure 3: Risks, Uncertainties and Bond Volatility

The solid line corresponds to the zscore of the first principal component of a bond volatility panel. This panel consists of the realized volatility of a 5-yr and 10-yr Treasury bond and the 5-yr and 10-yr TIV. The dotted red line depicts the first principal component of macroeconomic stochastic volatility ('risks'), extracted from a GARCH(1,1), applied to the one-quarter ahead SPF median inflation and GDP growth forecast. The green '-o' line reports the first principal component of a panel that consists for macroeconomic dispersion data ('Knightian uncertainty'), extracted from the cross-sectional volatility of one one-quarter ahead SPF inflation and GDP growth forecasts. The sample is from 1994:Q1 to 2009:Q2, the frequency is quarterly, and NBER recession dates are shaded.
Figure 4: State Variables of Model (Observed in Real-Time)

The upper panel plots the empirical proxies for economic activity ($z_t$) and monetary policy activity ($w_t$), annualized an in percent. The variable $z_t$ ($w_t$) coincides with the demeaned median one quarter ahead forecast for GDP growth (inflation). The bottom panel plots the observed amount of economic and monetary policy uncertainty. Economic uncertainty proxies for $\eta^r$ and is measured as the cross-sectional dispersion in one quarter ahead SPF GDP growth forecasts. Monetary policy uncertainty proxies for $\eta^m$ and is measured as the cross-sectional dispersion in one quarter ahead SPF inflation forecasts. The data frequency is quarterly and spans the period 1972:Q1 to 2009:Q2. The data in this picture is annualized and in percent. Yellow bars represent NBER recession dates.
The solid line represents model implied data points, whereas the dotted blue line corresponds to data plus 3 standard deviation confidence bounds. The upper left (right) panel presents the average term structure of realized volatility (option implied volatility, TIV) of bond returns for the time period 1994:Q1 to 2009:Q2. The lower left (right) panel presents the average term structure of realized volatility (option implied volatility, TIV) of bond returns for the time period 1972:Q1 to 2009:Q2. The maximum-likelihood estimation incorporates volatility data for the time period 1994:Q1 to 2009:Q2. The data on the y-axis is annualized and in percent, whereas data on the x-axis is in quarters.
This figure jointly plots the annualized average yield curve in the sample (with confidence error bounds) and the model implied yield curve. The maximum-likelihood estimation incorporates bond yield data for the time period 1972:Q1 to 2009:Q2 and uses only observable macro variables. The y-axis is in percent and the x-axis is in quarters.
The solid line represents model implied data points, whereas the dotted blue line corresponds to data. The left part of the picture (from top to bottom) represent: realized volatility, option implied volatility, and Treasury zero-coupon yield of a 4.5 year Macaulay duration bond. Analogously, the right part of the picture (from top to bottom) represent: realized volatility, option implied volatility, and Treasury zero-coupon yield of a 6.5 year Macaulay duration bond. The maximum-likelihood estimation incorporates volatility data for the time period 1994:Q1 to 2009:Q2 and Treasury yield data from 1972:Q1 to 2009:Q2. The data in this picture is annualized and in percent. Yellow bars represent NBER recession dates.
This figure presents the model implied term structure of physical volatility (upper panel) and option implied volatility (bottom panel) for the three periods: Great Moderation, Monetary Policy Experimentation, and the whole sample. The model is estimated over the whole sample and sample averages are used to construct the picture. The maximum-likelihood estimation incorporates volatility data for the time period 1994:Q1 to 2009:Q2 and Treasury yield data from 1972:Q1 to 2009:Q2. The data in this picture is annualized and in percent, the x-axis is in quarters.
This figure plots the model implied volatility on a one month option written on a 6.5-year Macaulay duration Treasury bond (solid line) and the observable real-time measure of Monetary Policy Uncertainty (dotted line) together with NBER recession dates. The maximum-likelihood estimation incorporates volatility data for the time period 1994:Q1 to 2009:Q2. The data in this picture is annualized and in percent. Yellow bars represent NBER recession dates.
This figure presents impulse responses. The driver of monetary policy uncertainty, $\eta^{MP}$, is shocked by one standard deviation and the response on various financial variables is analyzed. The financial variables of interest are (from top left to bottom right): physical volatility of a 6.5-year Macaulay duration bond, option implied volatility of a 6.5-year Macaulay duration bond, physical volatility of a 4.5-year Macaulay duration bond, option implied volatility of a 4.5-year Macaulay duration bond, slope the the Treasury yield curve. The $x$-axis is in quarters, numerical values on the $y$-axis are in percent and annualized. The maximum-likelihood estimation incorporates volatility data for the time period 1994:Q1 to 2009:Q2 and Treasury yield data from 1972:Q1 to 2009:Q2.