A Learning Model of Dividend Smoothing

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Abstract

We derive the optimal dynamic contract in a continuous-time principal-agent setting, in which both investors and the agent learn about the firm’s profitability over time. We show that the optimal contract can be implemented through the firm’s payout policy. The firm accumulates cash until it reaches a target balance that depends on the agent’s perceived productivity. Once this target balance is reached, the firm starts paying dividends equal to its expected future earnings, while any temporary shocks to earnings either add to or deplete the firm’s cash reserves. The firm is liquidated if its cash reserves fall below a minimum threshold. We also show that once the firm initiates dividends, this liquidation policy is first-best, despite the agency problem.

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One of the important puzzles in corporate finance is the smoothness of corporate dividends relative to earnings and cash flows. In an early empirical study, Lintner (1956) developed a model of dividend policy in which he proposed that firms adjust their dividends slowly to maintain a target long-run payout ratio. Specifically, Lintner argued that firms base their dividend decisions on their perception of the permanent component of earnings, and avoid adjusting dividends based on temporary or cyclical fluctuations. Numerous more recent studies have confirmed (e.g. Allen and Michaely (2003) and Brav et. al. (2005)) that firms engage in smoothing their dividends (and their payouts more generally). Furthermore, dividends tend to be paid by mature firms, and dividend changes tend to result in significant stock price reactions in the same direction, suggesting that investors view dividends as important indicators of the firm’s future cash flows.

In this paper we develop dynamic contracting model of dividend smoothing. We consider a natural principal-agent setting in which the agent can reduce effort in the firm and engage in outside activities that generate private benefits. Both the principal (outside investors) and the agent are risk neutral, but the agent is wealth-constrained. We depart from the standard principal-agent setting by assuming both investors and the agent learn over time about the firm’s expected future profitability based on its current cash flows. A contract provides the agent with incentives by specifying the agent’s compensation and whether the firm will continue or be forced to shut down as a function of the firm’s history of reported earnings.

After solving for the optimal contract, we show that it can be implemented through the firm’s payout policy and a capital structure in which the agent holds a share of the firm’s equity. When the firm is young, it makes no payouts and accumulates cash until it reaches a target level of financial slack that is positively related to the agent’s perceived productivity. Once this target balance is reached, the firm initiates dividend payments. From that point on, the firm pays dividends at a rate equal to its expected future earnings. The firm absorbs any temporary shocks to earnings by increasing or decreasing its cash

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1 When discussing payout policy we will often, for simplicity, refer to dividends alone, but our discussion should be interpreted to include share repurchases as well. Similarly, when discussing debt or leverage policies, we are referring to the firm’s net debt, which includes cash reserves.
reserves, and it may also borrow. However, when the firm’s debt reaches the liquidation value of its assets, the debt holders liquidate the firm.

This payout policy captures well the stylized facts associated with observed payout policies cited above. Immature firms do not pay dividends, but instead retain their earnings to invest, repay debt, and build cash reserves. For these firms the value of internal funds is high, as they risk running out of cash and being prematurely liquidated. But once the firm has sufficient financial slack, dividends are then paid. The level of dividends is based on the firm’s estimate of the permanent component of its earnings, resulting in dividend payments that are much smoother than its earnings. Because dividend changes reflect permanent changes to profitability, they are persistent and have substantial implications for firm value.

Despite the broad empirical evidence that firms smooth their dividends, normative theoretical models of dividend smoothing have proved rather elusive. Modigliani and Miller (1961) showed that dividend policy is irrelevant if capital markets are perfect and investment policy is held constant, so one could argue that observed dividend policy is one of many “neutral” variations that firms could adopt. Such a view is difficult to reconcile with the stock price reaction to dividends discussed above; it is also contrary to the large body of evidence that market imperfections are economically significant and important drivers of corporate financial policy more generally.

One key difficulty with providing a theoretical model in which firm’s optimally smooth their payouts relative to earnings is that, given a fixed investment policy, it necessarily implies that the firm’s leverage or cash position (its net debt) must be correspondingly “non-smooth.” This observation is difficult to reconcile with the standard trade-off theory of capital structure and payout policy, which predicts that firms will maintain a target level of leverage/financial slack with balances the tax benefits of leverage (equivalently, the tax disadvantage of retaining cash) with potential costs of financial distress. In such a model, temporary cash flow shocks should be passed through to the firm’s payouts as it tries to maintain its leverage target.

Myers’ (1984) description of the pecking order hypothesis does include the prediction that variations in net cash flow will be absorbed largely by adjustments to the firm’s debt.
However, this conclusion is based on the assumption that firms’ dividends are sticky in the short run, and no theoretical justification is provided for this assumption.²

We take a different approach in this paper. First, we use an optimal mechanism design approach to identify the real variables of interest in our model: the optimal timing of the liquidation decision, and the payoffs of the agent and investors after any history. In spirit of Modigliani and Miller (1961), in our model there may be many optimal dividend policies that can implement this optimal mechanism if there is no cost to raising equity capital in the event that the firm runs out of cash in the future.

Yet, in practice there are both institutional and informational costs to raising equity. Investment banks charge a 7% fee on public offerings. Moreover, new equity could be under-priced due to adverse selection. The pecking-order theory of Myers (1984) and the adverse selection arguments of Myers and Majluf (1984) imply that firms should raise funds using securities that are least sensitive to the firm’s private information, i.e. debt. These arguments suggest that firms should abstain from paying dividends in order to avoid future finance costs, at least as long as the risk of running out of cash and triggering inefficient liquidation exists.

With this in mind, we then identify the unique implementation of the optimal mechanism that provides the fastest possible payout rate subject to the constraint that the firm will not need to raise external capital in the future. Thus, in our model firms build up a target level of internal funds to ensure that they never need to liquidate inefficiently due to financial constraints. This constraint alone cannot explain dividend smoothing, as once the target is reached we would expect all excess cash flows to be paid out as dividends. The key driver of dividend smoothing in our model is the fact that there is learning about the firm’s profitability based on the current level of the firm’s cash flows. When cash flows are high, the firm’s perceived profitability increases. This raises the cost of liquidating the firm (we are liquidating a more profitable enterprise), and therefore raises the optimal level of financial slack. Thus, a portion of the firm’s high current cash flow

² Taken to its logical extreme, the adverse selection argument in favor of debt given by Myers and Majluf (1984) would suggest that the firm should never pay dividends in order to avoid future finance costs. If a firm were to pay dividends (for some other unknown reason), it would presumably be balancing the marginal benefit of dividends with the marginal value of financial slack. In this case one would expect dividends and financial slack to move together.
will optimally be used to increase its cash reserves, resulting in a smoothed dividend policy.

The firm pays out dividends *exactly* at the rate of expected earnings in our model in order to have just enough cash to avoid inefficient liquidation, given current profitability. Then the firm’s cash balances fall after bad news about future profitability due to negative earnings surprises, and rise after good news, due to positive earnings surprises. If the dividends were higher than expected earnings, the firm would run out of cash unless its profitability unexpectedly increases. If the firm’s dividends were lower than expected earnings, it would build more cash than necessary. This would lead to inefficiency if there are any costs of keeping funds inside the firm. Thus, dividend smoothing in our implementation of the optimal contract can be explained by two reasons (1) there are costs to raising outside equity capital and (2) there are slight costs to leaving extra cash in the firm.

Our model implies that firms smooth dividends by absorbing their cash flow variations through variations in their leverage or available financial slack. This conclusion is confirmed by empirical evidence. Indeed, as documented by Fazzari, Hubbard, and Petersen (1988) and others, financially unconstrained dividend-paying firms do appear to employ investment policies that are less sensitive to shocks to the firm’s earnings or cash flows. On the other hand, there is evidence that firms’ cash and leverage positions are strongly influenced by past profitability, even when firms are financially unconstrained (see, e.g., Fama and French (2002)).

In our model, the firm gains financial strength over time to mitigate the inefficiencies connected with moral hazard. This is a common prediction of a diverse range of dynamic contracting model, such as DeMarzo and Fishman (2003), Albuquerque and Hopenhayn (2004), Atkeson and Cole (2005), Clementi and Hopenhayn (2006) and DeMarzo and Sannikov (2006). The key ingredient of our setting that drives the smooth dividend policy is the combination of moral hazard and learning about the firm’s profitability. That is, it is important in our model that the firm’s cash flows carry information both about the agent’s effort and the agent’s skill.
In the next section of the paper, we describe the continuous-time principal-agent problem with learning about the firm’s profitability. Then in Section 2, we present the solution to this problem when moral hazard is absent, which is based purely on option-value considerations. This solution is important to understand the optimal long-term contract with moral hazard, which we derive in Section 3. Finally, in Section 4, we show how this optimal contract can be implemented in terms of the firm’s payout policy.

1. Model

In this section we present a continuous-time formulation of the firm and the principal-agent problem that arises between the manager running the firm and outside investors.

In our model, risk-neutral outside investors hire a risk-neutral agent to run a firm. Both the agent and investors discount the future at rate $r > 0$. Investors have unlimited wealth, whereas the agent has no initial wealth and must consume non-negatively.\(^3\)

The firm generates cash flows at an expected rate equal to $a_t \delta_t$, where $a_t \leq 1$ is the agent’s effort or activity level in the firm and $\delta_t$ is the expected profitability of the firm that depends on the agent’s managerial talent. These cash flows arrive with volatility $\sigma > 0$. Thus, the cumulative cash flow process $X_t$ for the firm is defined by

$$dX_t = a_t \delta_t \, dt + \sigma dZ_t$$

where $Z$ is a standard Brownian motion.

The firm’s profitability evolves over time, and the realization of the firm’s current cash flow is informative about these changes in profitability. We model this by assuming $\delta_t$ evolves according to

$$d\delta_t = \nu (dX_t - a_t \delta_t \, dt) = \nu \sigma dZ_t.$$

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\(^3\) The assumption that the agent has no initial wealth is without loss of generality; equivalently, we can assume the agent has already invested any initial wealth in the firm. The agent’s limited liability prevents a general solution to the moral hazard problem in which the firm is simply sold to the agent.
We remark that the specification in (1) and (2) is equivalent to the steady-state in a model with Bayesian updating in which $\delta_i$ is the conditional expectation of the firm’s unobserved true profitability, where there is an initial common prior that the firm’s true profitability is normally distributed with mean $\delta_0$ and is subject to mean zero, normally distributed additive shocks (which may or may not be correlated with the firm’s cash flows). In this case the posterior estimate of the firm’s profitability will evolve linearly with the surprises to the firm’s cash flows, as in (2).

The moral hazard problem stems from the fact that the agent’s activity within the firm is unobserved, and the agent can divert his own attention (or other resources) to outside opportunities. Given activity $a_t$ within the firm, the agent can engage in activity with effort $1-a_t$ outside the firm. These outside activities generate a private benefit for the agent at an expected rate $\lambda (1-a_t) \delta_t$, where the parameter $\lambda < 1$ reflects the fact that these outside activities are less productive than those engaged in within the firm.

The agent can also quit the firm at any time. In that event the agent devotes his full attention to these outside activities, earning private benefits at expected rate $\lambda \delta_t$. Thus, the value of the agent’s payoff in the event that he quits the firm is represented by

$$R(\delta_t) = R_0 + \frac{\lambda \delta_t}{r}$$

where $R_0$ captures any additional private benefits (or losses) associated with not being employed by the firm.

The firm requires external capital of $K \geq 0$ to be started. The investors contribute this capital and in exchange receive the cash flows generated by the firm less any compensation paid to the agent. The agent’s compensation is determined by a long-term contract. This contract specifies, based on the history of the firm’s cash flows, non-negative compensation for the agent while the firm operates, as well as a termination time when the firm is liquidated. Formally, a contract can be represent by a pair $(C, \tau)$, where $C$ is a non-decreasing $X$-measurable process that represent the agent’s cumulative compensation (i.e., $dC_t \geq 0$ is the agent’s compensation at time $t$) and $\tau$ is an $X$-
measurable stopping time. In the event that the firm is liquidated, we assume that investors receive a liquidation value of $L$ for the firm’s assets. The agent engages in his outside option and so receives the payoff specified in (3).

A contract $(C, \tau)$ together with an $\mathcal{X}$-measurable effort recommendation ($a$) is optimal if it maximizes the principal’s profit

$$E\left[\int_0^\tau e^{-rt}((1-a_t)\delta_t dt - dC_t) + e^{-r\tau}L\right]$$

subject to

$$W_0 = E\left[\int_0^\tau e^{-rt}(\lambda(1-a_t)\delta_t dt + dC_t) + e^{-r\tau}R(\delta_t)\right]$$ given strategy ($a$) (4)

and

$$W_0 \geq E\left[\int_0^\tau e^{-rt}(\lambda(1-\hat{a}_t)\delta_t dt + dC_t) + e^{-r\tau}R(\delta_t)\right]$$ for any other strategy ($a'$). (5)

By varying $W_0 > R(\delta_0)$, we can use this solution to consider different divisions of bargaining power between the agent and the investors. For example, if the agent enjoys all the bargaining power due to competition between investors, then the agent will receive the maximal value of $W_0$ subject to the constraint that the investors’ payoff be at least equal to their initial investment, $K$. We say that the effort recommendation ($a$) is incentive-compatible with respect to the contract $(C, \tau)$ if it satisfies the constraint (5).

Unlike in the classic principal-agent setting, in our contractual environment the agent may have private information not only about his effort, but also firm profitability. Indeed, if the principal knows the agent’s effort $\{a_t\}$, he can update his belief $\hat{\delta}_t$ about firm profitability according to

$$d\hat{\delta}_t = \nu(dx_t - a_t\hat{\delta}_t dt), \quad \hat{\delta}_0 = \delta_0.$$

However, if the agent deviates to a different effort strategy, the principal’s belief $\hat{\delta}_t$ becomes incorrect.
REMARKS. For simplicity, we specify the contract assuming that the agent’s compensation and the termination time $\tau$ are determined by the cash flow process, ruling out public randomization. This assumption is without loss of generality, as we will later verify that public randomization would not improve the contract.

2. The First-Best Solution.

Before we solve for the optimal contract in Section 3, let us derive the first-best solution. That is, we ignore the incentive constraints (5) and imagine the case when the principal can control the agent’s effort, ignoring the incentive constraints. Then it is optimal to let the agent take effort $a_t = 1$ until liquidation, since it is cheaper to provide the agent with a flow of utility by paying him than by letting him divert attention to private activities. Then the total cost of providing the agent with a payoff of $W_0$ is

$$E \left[ \int_0^\tau e^{-rt} dC_t \right] = W_0 - E \left[ e^{-r\tau} R(\delta_\tau) \right],$$

and the gross profit that the firm produces is

$$E \left[ \int_0^\tau e^{-rt} \delta_t \, dt + e^{-r\tau} L \right].$$

Thus, without moral hazard the principal must choose a stopping time $\tau$ that maximizes

$$E \left[ \int_0^\tau e^{-rt} \delta_t \, dt + e^{-r\tau} (L + R(\delta_\tau)) \right].$$

This is a standard real-option problem that can be solved by the methods of Dixit and Pindyck (1994). It is optimal to trigger liquidation when the expected profitability $\delta_\tau$ reaches a critical level of $\bar{\delta}$. Proposition 1 derives the principal’s profit and the optimal choice of $\bar{\delta}$ for this problem.

**Proposition 1.** Let $\bar{b}(\delta)$ be the solution to the ordinary differential equation
\[ r\bar{b}(\delta) = \delta + \frac{1}{2} \nu^2 \sigma^2 \bar{b}''(\delta) \]

on \([\delta, \infty)\), determined by boundary conditions \( \bar{b}(\delta) = L + R(\delta), \quad \bar{b}'(\delta) = R'(\delta), \) and \( \bar{b}(\delta) - \delta / r \to 0 \) as \( \delta \to 0 \). On \((-\infty, \delta]\), let \( \bar{b}(\delta) = R(\delta) \). Then for a contract that delivers value \( W_0 \) to an agent of type \( \delta_0 \), \( \bar{b}(\delta_0) - W_0 \) is the principal’s optimal profit in a setting without moral hazard. It is also an upper bound on the principal’s profit under moral hazard. This upper bound is attained if and only if the firm operates when \( \delta_t > \delta \), and it is liquidated when \( \delta_t \leq \delta \).

**Proof.** For an arbitrary contract \((C, r)\), consider the process

\[ G_t = e^{-rt} \bar{b}(\delta_t) + \int_0^t e^{-s} \delta_s ds. \]

Let us show that \( G_t \) is a submartingale. Using Ito’s lemma, the drift of \( G_t \) is

\[ -re^{-rt} \bar{b}(\delta_t) + e^{-rt} \frac{1}{2} \nu^2 \sigma^2 \bar{b}''(\delta_t) + e^{-rt} \delta_t, \]

which is equal to 0 when \( \delta_t > \delta \) and \( -re^{-rt}(L + R(\delta_t)) + e^{-rt} \delta_t < 0 \) when \( \delta_t < \delta \).

Therefore, the principal’s expected profit at time 0 is

\[
E \left[ e^{-rT} L + \int_0^T e^{-s} (\delta_s - C_s) ds \right] = E \left[ e^{-rT} (L + R(\delta_T)) + \int_0^T e^{-s} \delta_s ds - W_0 \right] \leq E[G] - W_0 \leq G_0 - W_0 = \bar{b}(\delta_0) - W_0.
\]
The inequalities above become equalities if and only if $\delta_r \leq \hat{\delta}$ and $\delta_t > \hat{\delta}$ before time $\tau$. QED

In our setting with moral hazard, if the agent had deep pockets, the first-best liquidation policy could be attained by letting the agent own the firm. Inefficiency may occur when the agent has limited wealth. The lower the agent’s wealth, the greater is the inefficiency. Since it is costless for the risk-neutral agent to postpone consumption when he discounts future at the market rate $r$, we expect that in the optimal contract under moral hazard the agent does not consume until the efficient outcome can be attained. Therefore, when the agent starts consuming, the principal’s profit and the liquidation policy must be given by Proposition 1. We use Proposition 1 in the next section, where we derive the optimal contract under moral hazard.

3. Deriving the Optimal Contract.

The task of finding the optimal dynamic contract in a setting like ours is complex, because there is a huge space of fully contingent history-dependent contracts to consider. A contract $(C, \tau)$ must specify how the agent’s consumption and the liquidation time depend on the entire history of cash flows. In classic settings with uncertainty only about the agent’s effort but not the agent’s skill, there are standard recursive methods to deal with such complexity of contracts. These methods rely on dynamic programming using the agent’s future expected payoff (a.k.a. continuation value) as a state variable.\(^4\)

With uncertainty about the agent’s skill, standard methods do not apply directly to our model. Solving such a problem involves creativity. While the specific solution is unique to our problem, the general approach to such models involves three steps. First, one formulates necessary conditions for incentive-compatibility, which include all the binding constraints of the original problem. Second, one solves the relaxed optimization problem

\(^4\) For example, see Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) (in discrete time) and DeMarzo and Sannikov (2006) and Sannikov (2007a) (in continuous time) for the development of these methods, and Piskorski and Tchistyi (2006) and Philippon and Sannikov (2007) for their applications.
under a limited set of constrains identified in the first step. Lastly, one verifies that the solution of the relaxed problem is fully incentive-compatible.

The necessary incentive-compatibility constraints are formulated using appropriately chosen state variables. For our problem, we must include as state variables at least the principal’s current belief about the agent’s skill \( \delta_t \), which evolves according to

\[
d\delta_t = \nu(dX_t - a_t \delta_t)dt, \quad \delta_0 = \delta_0,
\]

and the agent’s continuation value (when the agent follows the recommended strategy \( \{a_s = 1\} \) after time \( t \), and the principal has a correct belief about the agent’s skill)

\[
W_t = E_t \left[ \int_t^\tau e^{-r(s-t)}dC_s + e^{-r(\tau-t)}R(\delta_\tau) \mid \delta_t = \delta_t \right].
\]

The variables \( \delta_t \) and \( W_t \) are well-defined for any contract \( (C, \tau) \), after any history of cash flows \( \{ X_s, s \in [0, t]\} \). However, they do not fully summarize the agent’s incentives, which depend on the agent’s deviation payoffs, the payoff that the agent would obtain if \( \delta_t' \neq \delta_t \) due to the agent’s past deviations. Therefore, we can formulate only necessary conditions for incentive compatibility using the variables \( \delta_t \) and \( W_t \).

The remainder of our derivation proceeds as follows. Lemma 1 provides a stochastic representation for the dependence of \( W_t \) on the cash flows \( \{X_t\} \) in a given contract \( (C, \tau) \). The connection between \( W_t \) and \( X_t \) matters for the agent’s incentives. Lemma 2 presents a necessary condition for incentive compatibility of the agent’s effort strategy \( \{a_t=1, t \leq \tau\} \). Subsection 3.1 conjectures an optimal contract and verifies that it is fully incentive-compatible. Subsection 3.2 verifies the optimality of the conjectured contract, by showing that it attains expected profit that is at least as high as any contract that satisfies the necessary conditions of Lemma 2.

The following representation of \( W_t \) is standard in continuous-time contracting:

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5 For example, in continuous time Sannikov (2007b) solves an agency problem with adverse selection using the continuation values of the two types of agents as state variables. Williams (2007) solves an example with hidden savings using the agent’s continuation value and his marginal utility as state variables.
Lemma 1. There exists a process $\{\beta_t, t \geq 0\}$ in $L^*$ such that

$$dW_t = rW_t dt - dC_t + \beta_t (dX_t - \hat{\delta}_t dt). \tag{6}$$

Proof. See Appendix.

Lemma 2 provides a necessary condition for a contract $(C, \tau)$ to be incentive compatible.

Lemma 2. A contract $(C, \tau)$ in which the agent’s continuation value evolves according to (6) is not incentive-compatible unless $\beta_t \geq \lambda(1 + \nu/r)$ for all $t \geq 0$.

Proof. Suppose that $\beta_t < \lambda(1 + \nu/r)$ on a set of positive measure. Let us show that the agent has a strategy $\{a_t, t \geq 0\}$ that allows him to attain an expected payoff greater than $W_0$.

Let $a_t = \frac{\hat{\delta}_t}{\delta_t}$ when $\beta_t \geq \lambda(1 + \nu/r)$ and $a_t = \frac{\hat{\delta}_t - 1}{\delta_t}$ when $\beta_t < \lambda(1 + \nu/r)$. Define the process

$$V_t = e^{-r t} \left( W_t + (\hat{\delta}_t - \hat{\delta}_t) \frac{\lambda}{r} \right) + \int_0^t e^{-r s} (dC_s + \lambda(1 - a_s) \delta_s ds).$$

If the agent follows the strategy described above, then

$$d\delta_t - d\hat{\delta}_t = \nu(\hat{\delta}_t - a_t \delta_t) dt \Rightarrow \delta_t \geq \hat{\delta}_t \text{ for all } t, \text{ and}$$

$$dW_t = rW_t dt - dC_t + \beta_t (a_t \delta_t dt + \sigma dZ_t - \hat{\delta}_t dt).$$

Therefore,

$$\frac{dV_t}{e^{-r t}} = -r \left( W_t + (\delta_t - \hat{\delta}_t) \frac{\lambda}{r} \right) dt + \left( rW_t dt - dC_t + (\beta_t - \nu \frac{\lambda}{r}) (a_t \delta_t - \hat{\delta}_t) dt + \sigma dZ_t \right) + dC_t + \lambda(1 - a_t) \delta_t dt$$

$$= (\lambda + \nu \frac{\lambda}{r} - \beta_t) (\hat{\delta}_t - a_t \delta_t) dt + \sigma dZ_t.$$
The drift of $V_t$ is everywhere nonnegative, and positive on a set of positive measure. Since $R(\delta_t) = \left( W_t + (\delta_t - \delta_t) \frac{\lambda}{r} \right)$, the agent’s payoff from this strategy

$$E \left[ e^{-\tau} R(\delta_t) + \int_0^\tau e^{-s} (dC_s + \lambda(1-a_s)\delta_s ds) \right] = E[V_{\tau}] > V_0 = W_0.$$

We conclude that $\beta_t \geq \lambda(1 + \nu/r)$ for all $t \geq 0$ is a necessary condition for the incentive compatibility of the agent’s strategy. QED

3.1. The Optimal Contract

In this section we conjecture the form of the optimal contract that delivers value $W_0 \geq R(\delta_0)$ to an agent of type $\delta_0 \geq \delta_t$. The contract is based on two state variables: the agent’s perceived skill level $\delta_t$ and his continuation value $W_t$. The evolution of $\hat{\delta}_t$ is not a contractual choice; it is determined by

$$d\hat{\delta}_t = \nu(dX_t - a_t\hat{\delta}_t dt), \quad \hat{\delta}_0 = \delta_0.$$

To describe how $W_t$ evolves, define

$$W^l(\delta) = R(\delta) + (\lambda/\nu + \lambda/r)(\delta - \delta).$$

If $W_0 < W^l(\delta_0)$, let $W_t$ evolve according to

$$dW_t = rW_t dt + \lambda(1+\nu/r)(dX_t - \hat{\delta}_t dt),$$

with $C_t = 0$, until $W_t$ reaches $R(\hat{\delta})$, resulting in termination, or $W^l(\hat{\delta})$. If $W_0 > W^l(\delta_0)$, let the principal make an immediate payment to the agent of $C_0 = W_0 - W^l(\delta_0)$, to let $W_0$ drop to $W^l(\delta_0)$. Once $W_t$ reaches $W^l(\delta_t)$ at time $\tau$, let the principal make payments to the agent at rate $dC_s = rW_s ds$ for all $s \geq \tau$, so that $W_s$ evolves according to

$$dW_s = \lambda(1+\nu/r)(dX_s - \hat{\delta}_s ds), \quad s \geq \tau.$$

Then $W_s = W^l(\delta_s)$ for all times $s \geq \tau$, until $\delta_s$ reaches the level $\delta$ and $W_s$ reaches $R(\delta) = R(\delta_0)$ at time $\tau$. By Proposition 1, this contract achieves first-best profit $W_0 \geq W^l(\delta_0)$.
We verify that this contract is incentive-compatible in this subsection, and justify its optimality in subsection 3.2. We cannot rely on Lemma 2 to check that the agent’s strategy \( \{a_t = 1\} \) maximizes his payoff, since Lemma 2 only gives a necessary condition. The following proposition verifies that the conjectured contract is incentive-compatible from the first principles.

**Proposition 2.** Under the contract outlined above, the agent of skill level \( \delta_0 \) gets the same value \( W_0 \) independently of his strategy. In particular, by choosing \( a_t = 1 \) at all times \( t \), the agent maximizes his payoff.

**Proof.** For any strategy of the agent, the process

\[
V_t = e^{-\gamma t} \left( W_t + (\delta_t - \delta_t) \frac{\lambda}{r} \right) + \int_0^t e^{-\gamma s} (dC_s + \lambda (1 - a_s) \delta_s) ds.
\]

defined in the proof of Lemma 2 has drift

\[
e^{-\gamma} (\lambda + \nu \frac{\lambda}{r} - \beta) (\delta_t - a \delta_t) = 0.
\]

Therefore, the agent’s expected payoff equals

\[
E \left[ e^{-\gamma t} R(\delta_t) + \int_0^t e^{-\gamma s} (dC_s + \lambda (1 - a_s) \delta_s) ds \right] = E[V_t] = V_0 = W_0
\]

independently of the agent’s strategy. QED

**3.2. Justification of the Optimal Contract.**

In this section we verify that the contract presented in subsection 3.1 is optimal. We will show that this contract attains the highest expected profit among all contracts that deliver value \( W_0 \) to an agent of skill level \( \delta_0 \) and satisfy the necessary incentive-compatibility condition of Lemma 2. The set of such contracts includes all fully incentive-compatible contracts. Since the contract of subsection 3.1 is incentive-compatible, as shown in Proposition 2, it follows that it is also optimal.
Let us present a roadmap of our verification argument. First, we define a function \( b(W_0, \delta_0) \), which gives the expected profit that a contract of subsection 3.1 attains for any pair \((W_0, \delta_0)\) with \( W_0 \geq R(\delta) \) and \( \delta_0 \geq \delta \). Proposition 3 verifies that this is indeed true for our definition of \( b(W_0, \delta_0) \). After that, Proposition 4 shows that the principal’s profit in any alternative contract that satisfies the necessary incentive-compatibility condition of Lemma 2 is at most \( b(W_0, \delta_0) \) for any pair \((W_0, \delta_0)\) with \( W_0 \geq R(\delta) \) and \( \delta_0 \geq \delta \). It follows that the contract of subsection 3.1 is optimal.

For \( W \geq R(\delta) \) and \( \delta \geq \delta \), define a function \( b(W, \delta) \) as follows.

(i) For \( W > W_1(\delta) \), let \( b(W, \delta) = \bar{b}(\delta) - W \).

(ii) For \( W = R(\delta) \), let \( b(W, \delta) = L \).

Otherwise, for \( \delta > \delta \) and \( W \in (R(\delta), W_1(\delta)) \), let \( b(W, \delta) \) solve the equation

\[
rb(W, \delta) = \delta + rWb_w(W, \delta) + \frac{1}{2} \lambda^2 (1 + \frac{\nu}{\sigma})^2 b_{ww}(W, \delta) + \frac{1}{2} \nu^2 \sigma^2 b_{\delta\delta}(W, \delta) + \lambda (1 + \frac{\nu}{\sigma}) \nu \sigma^2 b_{w\delta}(W, \delta)
\]

with boundary conditions given by (i) and (ii).

For an arbitrary incentive-compatible contract \((C, \tau)\) that motivates the agent to put effort \( \{a_t = 1\} \), in which the agent’s continuation value evolves according to (6), define the process

\[
G_t = \int_0^t e^{-r\sigma} (\delta_s ds - dC_s) + e^{-r\tau} b(\delta_t, W_t).
\]

Note that \( \delta_t = \hat{\delta}_t \) for any incentive-compatible contract.

Lemma 3 helps us prove both Propositions 3 and 4.

**Lemma 3.** When \( \delta_t \geq \delta \) and \( C_t \) is continuous at \( t \), then

\[
dG_t = e^{-r\sigma} (vb_{\delta}(W_t, \delta_t) + \lambda (1 + \frac{\nu}{\sigma}) b_w(W_t, \delta_t)) \sigma dZ_t - e^{-r\tau} (b_w(W_t, \delta_t) + 1) dC_t + e^{-r\sigma} \left( \frac{1}{2} (\beta_t - \lambda (1 + \frac{\nu}{\sigma}))^2 b_{ww}(W_t, \delta_t) + (\beta_t - \lambda (1 + \frac{\nu}{\sigma}))(\lambda (1 + \frac{\nu}{\sigma}) b_{w\delta}(W_t, \delta_t) + \nu b_{w\delta}(W_t, \delta_t)) \right) dt
\]
Proof. See Appendix A.

**Proposition 3.** The conjectured optimal contract of subsection 3.1 attains profit \( b(\delta_0, W_0) \).

**Proof.** Under that contract, the process \( G_t \) is a martingale. Indeed, for all \( t > 0 \), the continuous process \( C_t \) increases only when \( W_t = W^d(\delta_t) \) (where \( b_W(W^d(\delta_t), \delta_t) = -1 \)) and \( \beta_t = \lambda(1+\frac{c}{\tau}) \), so \( G_t \) is a martingale by Lemma 3. At time 0, the agent consumes positively only in order for \( W_0 \) to drop to \( W^d(\delta_0) \), and \( b_W(W, \delta_0) = -1 \) for \( W \geq W^d(\delta_0) \), so \( G_t \) is a martingale there as well. Therefore, the principal attains the profit of

\[
E \left[ e^{-rt} b(W_t, \delta_t) + \int_0^t e^{-rt} (\delta_t dt - dC_t) \right] = E[ G_t ] = G_0 = b(W_0, \delta_0).
\]

QED.

**Proposition 4.** In any alternative incentive-compatible contract \((C, \tau)\) the principal’s profit is bounded from above by \( b(W_0, \delta_0) \).

**Proof.** Let us argue that \( G_t \) is a supermartingale for any alternative incentive-compatible contract \((C, \tau)\) while \( \delta_t \geq \delta \). Appendix B shows that

\[
b_{WW}(W, \delta) \leq 0 \quad \text{and} \quad \lambda(1+\frac{V}{\rho})b_{WW}(W, \delta) + \nu b_{W,\delta}(W, \delta) \leq 0
\]

for all pairs \((W, \delta)\), so that also \( b_W(W, \delta) \geq -1 \) for all \((W, \delta)\). It follows that whenever \( C_t \) has an upward jump of \( \Delta C_t \), \( G_t \) has a jump of \( e^{-rt} (b(W_t+\Delta C_t, \delta_t) - b(W_t, \delta_t) - \Delta C_t) \leq 0 \). Whenever \( C_t \) is continuous, \( G_t \) is a supermartingale by Lemma 3, while \( \delta_t \geq \delta \).

Now, let \( \tau \) be the earlier of the liquidation time or the time when \( \delta_t \) reaches the level \( \delta \). Then Proposition 1 implies that the principal’s profit at time \( \tau \) is bounded from above by \( b(W_{\tau}, \delta_{\tau}) \). It follows that the principal’s total expected profit is bounded from above by
\[
E \left[ e^{-rT} b(W_0, \delta_0) + \int_0^T e^{-r} (\delta_t dt - dC_t) \right] = E \left[ G_0 \right] \leq G_0 = b(W_0, \delta_0).
\]

QED

We conclude that subsection 3.1 presents the optimal incentive-compatible contract for any pair \((W_0, \delta_0)\) such that \(W_0 \geq R(\delta_0)\) and \(\delta_0 \geq \delta\). If \(W_0 \geq W^d(\delta_0)\), then this contract attains the first-best profit, and liquidation always occurs at the efficient level of profitability of \(\delta_t = \delta\). If \(W_0 < W^d(\delta_0)\), then liquidation happens inefficiently with positive probability.

In Section 4, we suggest one capital structure implementation of the optimal contract. Rather than defining the agent’s consumption and the liquidation time in terms of abstract variables \(\delta_t\) and \(W_t\), our implementation gives the agent a fraction of the firm’s equity, and allows him to control the firm’s payout policy. The firm may use available cash balances or borrowing up to the liquidation value of the assets \(L\) to offset operating losses and pay out dividends. Default occurs when the firm’s debt reaches \(L\). Under the capital structure we propose, by optimally using his discretion, the agent implements the same outcome as the contract of subsection 3.1.


In this section we implement the optimal contract using a remarkably simple capital structure. Let the agent hold a fraction \(\lambda\) of the firm’s equity, and have full discretion over the firm’s dividend policy. Dividends can be paid out of the firm’s cash balances, which are also used to absorb gains and losses of cash flows. The starting cash level in the firm is determined by

\[
M_o = \frac{W_0}{\lambda} - \frac{\delta_0}{r},
\]

and the firm is liquidated if the cash balances drop to a critical level. Proposition 5 shows that if this critical level is given by \(R_0/\lambda\), then the agent has incentives to follow a
dividend policy that attains the same outcome as the optimal contract of the previous section.

Note that the cash balance $M_t$ inside the firm evolves according to

$$dM_t = rM_t dt - dD_t + dX_t$$

under this capital structure implementation, where are the dividends paid out by the agent, of which the agent receives $dC_t = \lambda dD_t$.

**Proposition 5.** Under the capital structure presented above, the agent’s future expected payoff at time $t$ is given by

$$W_t = \lambda(M_t + \delta_t/r).$$

(9)

The agent has incentives to choose $a_t = 1$ at all times, to pay no dividends when $W_t < W^d(\delta)$, and to pay them out at rate $dD_t = rW_t/\lambda dt$ when $W_t = W^d(\delta)$. By doing so, the agent implements the optimal contract.

The proof in Appendix C shows that the agent is in fact indifferent between all effort strategies and dividend policies. If the agent chooses to reduce the cash balance to the critical level immediately, he gets the value of

$$W_t = \lambda(M_t - R_0/\lambda) + R(\delta) = \lambda(M_t + \delta_t/r).$$

If the agent chooses to run the firm without paying dividends, the firm produces earnings at the expected rate of $rM_t + \delta$. A fraction $\lambda$ of the firm’s earnings compensates the agent just enough for postponing the value of $W_t$, while the expected earnings $\delta_t$ follow a martingale. This makes the agent just indifferent between “cashing out” and running the firm. The agent may also divert attention to private activities to get a benefit of $\lambda$ for each dollar loss of the firm’s cash flows. Since the agent can also convert one dollar of output into $\lambda$ units of consumption by paying dividends, he is indifferent between all effort levels.
If the agent refrains from paying dividends while \( W_t < W^t(\delta_t) \) and pays dividends at the rate of expected earnings \( rM_t + \delta_t \) when \( W_t = W^t(\delta_t) \), he implements the optimal contract, since then \( W_t \) evolves in exactly the same way as in the optimal contract of subsection 3.1, and liquidation happens when \( W_t \) reaches the level \( \lambda(R_\delta/\lambda + \delta_t/r) = R(\delta) \) (the same as in Section 3).

It is remarkable that under this implementation, the set of options available to the agent depends neither on the firm’s perceived profitability \( \hat{\delta}_t \) nor on the agent’s “continuation value” \( W_t \). The optimal time of liquidation is determined by a simple variable, the firm’s cash balance \( M_t \) and the critical threshold \( R_\delta/\lambda \).

5. Appendix A.

**Proof of Lemma 1.** Note that

\[
V_t = \int_0^t e^{-\gamma s} dC_s + e^{-\gamma t} W_t
\]
is a martingale when the agent follows the strategy \( \{a_s = 1\} \). By the Martingale Representation Theorem there exists a process \( \{\beta_t, t \geq 0\} \) in \( L^* \) such that

\[
V_t = V_0 + \int_0^t e^{-\gamma s} \beta_s (dX_s - \hat{\delta}_t ds),
\]
since \( dX_s - \hat{\delta}_t ds = \sigma dZ_s \) when \( a_s = 1 \). Differentiating with respect to \( t \), we find that

\[
dV_t = e^{-\gamma t} dC_t + e^{-\gamma t} dW_t - re^{-\gamma t} W_t dt = e^{-\gamma t} \beta_t (dX_t - \hat{\delta}_t dt) \Rightarrow
\]

\[
dW_t = rW_t dt - dC_t + \beta_t (dX_t - \hat{\delta}_t dt).
\]

QED.

**Proof of Lemma 3.** Note that for \( \delta \geq \hat{\delta} \), the function \( b \) satisfies partial differential equation (7) even if \( W > W^t(\delta) \). Indeed, since \( b(W, \delta) = \bar{b}(\delta) - W \) and \( b_W = -1 \) in that region, the equation reduces to
\[ r(\bar{b}(\delta) - W) = \delta - rW + \frac{1}{2} \nu^2 \sigma^2 \bar{b}''(\delta). \]

This equation holds by the definition of \( \bar{b} \).

When \( C_t \) is continuous at \( t \), then using Ito’s lemma,

\[
db(W_t, \delta_t) = \left( rW_t dt - dC_t \right) b_{W}(W_t, \delta_t) + \sigma^2 \left( \frac{1}{2} \beta_{W}^2 b_{W}(W_t, \delta_t) + \nu^2 b_{W, \delta}(W_t, \delta_t) + \beta b_{W, \delta}(W_t, \delta_t) \right) dt \\
+ \left( \nu b_{W}(W_t, \delta_t) + \beta b_{W}(W_t, \delta_t) \right) \sigma dZ_t = rb(W_t, \delta_t) dt - \delta_t dt - b_{W}(W_t, \delta_t) dC_t \\
+ \sigma^2 \left( \frac{1}{2} (\beta - \lambda(1 + \frac{\nu}{r}))^2 b_{W_{\nu, \delta}}(W_t, \delta_t) dt + (\beta_t - \lambda(1 + \frac{\nu}{r}))(\lambda(1 + \frac{\nu}{r})b_{W_{\nu, \delta}}(W_t, \delta_t) + \nu b_{W_{\nu, \delta}}(W_t, \delta_t)) \right) dt,
\]

where the second equality follows from (7). From the definition of \( G_t \), it follows that Lemma 3 correctly specifies how \( G_t \) evolves. QED

6. Appendix B.

We must show that for all pairs \((\delta, W)\), the function \( b(\delta, W) \) satisfies

\[ b_{W_{\nu, \delta}}(W, \delta) \leq 0 \text{ and } \lambda(1 + \frac{\nu}{r})b_{W_{\nu, \delta}}(W, \delta) + \nu b_{W_{\nu, \delta}}(W, \delta) \leq 0. \]

It is useful to understand the dynamics of the pair \((\delta_t, W_t)\) under a conjectured optimal contract first. The pair \((\delta_t, W_t)\) follows

\[ d\delta_t = \nu \sigma dZ_t \text{ and } dW_t = rW_t dt + \lambda(1 + \nu r) \sigma dZ_t \text{ until } W_t \text{ reaches } W^f(\delta_t), \]

and \( W_t = W^f(\delta_t) \) thereafter. (A1)

When \( W_t \) reaches the level \( R(\delta_t) \), termination results. The lines parallel to \( W^f(\delta) \) are the paths of the joint volatilities of \((W_t, \delta_t)\). Due to the positive drift of \( W_t \), the pair \((W_t, \delta_t)\) moves across these lines in the direction of increasing \( W_t \). See the figure below for reference.
The phase diagram of \((W_t, \delta_t)\) provides two important directions: the direction of joint volatilities, in which \(dW/d\delta = \lambda(1 + \frac{V}{r})/\nu\), and the direction of drifts, in which \(W\) increases but \(\delta\) stays the same. We need to prove that \(b_{W}(\delta, W)\) weakly decreases in both of these directions.

To study how \(b_{W}(W, \delta)\) depends on \((W, \delta)\), it is useful to know that \(b_{W}(W_t, \delta_t)\) is a martingale (Lemma 4) and that \(b_{W}(R(\delta), \delta)\) increases in \(\delta\) (Lemma 5).

**Lemma 4.** When the evolution of \((W_t, \delta_t)\) is given by (A1), then \(b_{W}(W_t, \delta_t)\) is a martingale.

**Proof.** Differentiating the partial differential equation for \(b(W, \delta)\) with respect to \(W\), we obtain

\[
0 = rWb_{WW}(W, \delta) + \frac{1}{2} \lambda^2 (1 + \frac{V}{r})^2 \sigma^2 b_{WWW}(W, \delta) + \frac{1}{2} \nu^2 \sigma^2 b_{W\delta \delta}(W, \delta) + \lambda(1 + \frac{V}{r}) \nu \sigma^2 b_{W\delta \delta}(W, \delta).
\]

The right hand side of this equation is the drift of \(b_{W}(W, \delta)\) when \(W_t < W^d(\delta)\) by Ito's lemma. When \(W_t = W^d(\delta)\), then \(b_{W}(W_t, \delta_t) = -1\) at all times. Therefore, \(b_{W}(W_t, \delta_t)\) is always a martingale. QED
Lemma 5. \( b_w(R(\delta), \delta) \) weakly increases in \( \delta \).

Proof. Note that

\[
b(W_0, \delta_0) = \bar{b}(\delta_0) - W_0 + E\left[ e^{-r_T} (L - \bar{b}(\delta_T) + R(\delta_T)) \mid \delta_0, W_0 \right],
\]

where \( \bar{b}(\delta_T) - L \) is the loss of value relative to first-best due to premature liquidation.

Let us show that for all \( \varepsilon > 0 \), \( b(R(\delta_0) + \varepsilon, \delta_0) - b(R(\delta_0), \delta_0) \) increases in \( \delta_0 \). Note that

\[
b(R(\delta_0) + \varepsilon, \delta_0) - b(R(\delta_0), \delta_0) =
\]

\[
\bar{b}(\delta_0) - R(\delta_0) - \varepsilon + E\left[ e^{-r_T} (L - \bar{b}(\delta_T) + R(\delta_T)) \mid \delta_0, R(\delta_0) + \varepsilon \right] - L =
\]

\[
E\left[ e^{-r_T} (\bar{b}(\delta_0) - R(\delta_0) - \bar{b}(\delta_T) + R(\delta_T)) + (1 - e^{-r_T})(\bar{b}(\delta_0) - R(\delta_0) - L) \mid \delta_0, R(\delta_0) + \varepsilon \right] - \varepsilon
\]

Consider the processes \((W^i_t, \delta^i_t) (i = 1,2)\) that follow (A1) starting from values \((W^i_0, \delta^i_0) = (R(\delta^i_0), \delta^i_0)\) such that \( \delta^2_0 - \delta^1_0 = \Delta > 0 \). Then for any path of \( Z \), the difference \( \delta^2_t - \delta^1_t \) stays equal to \( \Delta \) and \( W^2_t - W^1_t \) becomes larger than \( \lambda / r \Delta \) for all \( t > 0 \). Therefore,

(i) at the time \( \tau_1 \) when \( W^1_t \) reaches level \( R(\delta^1_t) \), we still have \( W^2_t > R(\delta^2_t) \), and

(ii) at time \( \tau_2 > \tau_1 \) when \( W^2_t \) reaches \( R(\delta^2_t) \), we have \( \delta^2_{\tau_2} < \delta^1_{\tau_1} + \Delta \).

Thus,

\[
b(R(\delta^2_0) + \varepsilon, \delta^2_0) - b(R(\delta^2_0), \delta^2_0) =
\]

\[
E\left[ e^{-r_{\tau_2}} (\bar{b}(\delta^2_0) - R(\delta^2_0) - \bar{b}(\delta^2_{\tau_1}) + R(\delta^2_{\tau_1})) + (1 - e^{-r_{\tau_2}})(\bar{b}(\delta^2_0) - R(\delta^2_0) - L) \mid \delta_0, R(\delta_0) + \varepsilon \right] - \varepsilon \geq
\]

\[
E\left[ e^{-r_{\tau_2}} (\bar{b}(\delta^2_0) - R(\delta^2_0) - \bar{b}(\delta^2_{\tau_1}) + R(\delta^2_{\tau_1})) + (1 - e^{-r_{\tau_2}})(\bar{b}(\delta^2_0) - R(\delta^2_0) - L) \mid \delta_0, R(\delta_0) + \varepsilon \right] - \varepsilon \geq
\]

\[
(1 - e^{-r_{\tau_2}})(\bar{b}(\delta^2_0) - R(\delta^2_0) - L) \mid \delta_0, R(\delta_0) + \varepsilon \] - \varepsilon =
\]

\[
b(R(\delta^1_0) + \varepsilon, \delta^1_0) - b(R(\delta^1_0), \delta^1_0),
\]

since

\[
\bar{b}(\delta^2_0) - R(\delta^2_0) - \bar{b}(\delta^2_{\tau_1}) + R(\delta^2_{\tau_1}) \geq \bar{b}(\delta^2_0) - R(\delta^2_0) - \bar{b}(\delta^2_{\tau_1} + \Delta) + R(\delta^2_{\tau_1} + \Delta) =
\]

\[
\bar{b}(\delta^1_0 + \Delta) - R(\delta^1_0 + \Delta) - \bar{b}(\delta^2_{\tau_1} + \Delta) + R(\delta^2_{\tau_1} + \Delta) \geq \bar{b}(\delta^1_0) - R(\delta^1_0) - \bar{b}(\delta^1_{\tau_1}) + R(\delta^1_{\tau_1}),
\]

\[
\bar{b}(\delta^2_0) - R(\delta^2_0) - L \geq \bar{b}(\delta^1_0) - R(\delta^1_0) - \bar{b}(\delta^1_{\tau_1}) + R(\delta^1_{\tau_1}) \quad \text{and}
\]
We can use Lemmas 4 and 5 to reach conclusions about how $b_w(W, \delta)$ changes as $W$ increases or as $\delta$ and $W$ increase in the direction $dW/d\delta = \lambda(1+\frac{\nu}{r})/\nu$.

**Lemma 6.** $b_w(W, \delta)$ weakly decreases in $W$.

**Proof.** Let us show that for any $\delta_0 \geq \delta$, for any two values $W_0^1 < W_0^2$, $b_w(W_t^1, \delta_0) \geq b_w(W_t^2, \delta_0)$.

Consider the processes $(W_t^i, \delta_t^i) (i = 1,2)$ that follow (A1) starting from values $(W_0^1, \delta_0)$ and $(W_0^2, \delta_0)$ for $\delta_0^i < \delta_0^2$. Then for any path of $Z$, we have $W_t^2 - W_t^1 \geq 0$ until time $\tau_1$ when $W_t^1$ reaches the level of $R(\delta)$. The time when $W_t^2$ reaches the level of $R(\delta)$ is $\tau_2 \geq \tau_1$.

Since $W_{\tau_1}^2 \geq W_{\tau_1}^1 = R(\delta_{\tau_1})$, it follows that $\delta_{\tau_2} \leq \delta_{\tau_1}$ and $W_{\tau_2}^2 \leq W_{\tau_1}^1$. Using Lemmas 4 and 5,

$$b_w(W_0^1, \delta_0) = E\left[b_w(R(\delta_{\tau_1}), \delta_{\tau_1})\right] \geq E\left[b_w(R(\delta_{\tau_2}), \delta_{\tau_2})\right] = b_w(W_0^2, \delta_0).$$

QED

**Lemma 7.** $b_w(W, \delta)$ weakly decreases in the direction, in which $W$ and $\delta$ increase according to $dW/d\delta = \lambda(1+\frac{\nu}{r})/\nu$.

**Proof.** Consider starting values $(W_0^i, \delta_0^i)$ that satisfy

$$\delta_0^2 - \delta_0^1 = \Delta > 0 \text{ and } W_0^2 - W_0^1 = \Delta \lambda(1+\frac{\nu}{r})/\nu.$$

Starting from those values, let the processes $(W_t^i, \delta_t^i) (i = 1,2)$ follow (*). Then for any path of $Z$, at all times $\delta_t^2 - \delta_t^1 = \Delta$ and $W_t^2 - W_t^1 \geq \Delta \lambda(1+\frac{\nu}{r})/\nu$ (with equality after time 0 only if $W_t^2 = W_t^1 = \Delta \lambda(1+\frac{\nu}{r})/\nu$). Therefore, the time $\tau_i$ when $W_t^i$ reaches the level of $R(\delta_t^i)$ occurs at least as soon as the time $\tau_2$ when $W_t^2$ reaches the level of $R(\delta_t^2)$.

Also, since $W_{\tau_1}^2 \geq W_{\tau_1}^1 + \Delta \lambda(1+\frac{\nu}{r})/\nu > R(\delta_{\tau_1}) + \Delta \lambda/r$, it follows that $\delta_{\tau_2} \leq \delta_{\tau_1}$ and $W_{\tau_2}^2 \leq W_{\tau_1}^1$. Using Lemmas 4 and 5,

$$b_w(W_0^1, \delta_0) = E\left[b_w(R(\delta_{\tau_1}^1), \delta_{\tau_1}^1)\right] \geq E\left[b_w(R(\delta_{\tau_2}^2), \delta_{\tau_2}^2)\right] = b_w(W_0^2, \delta_0^2).$$
7. Appendix C.

Proof of Proposition 5. The variable $W_t$ defined by (9) evolves according to

$$dW_t = \lambda (rM_t dt - dD_t + dX_t) + \lambda/r \nu (dX_t - a_t \delta_t dt)$$

$$= r(W_t - \lambda \delta_t/r) dt - dC_t + \lambda \nu/r (dX_t - a_t \delta_t dt)$$

$$= rW_t dt - dC_t + \lambda (1 + \nu/r) (dX_t - a_t \delta_t dt) - \lambda (1 - a_t) \delta_t dt.$$

Let us show that $W_t$ is the agent’s future expected payoff for any strategy for which the transversality condition $E[e^{-r}W_t] \to 0$ as $t \to \infty$ holds (it does for the strategy prescribed in the proposition). Then the process

$$V_t = e^{-r}W_t dt + \int_0^t e^{-r} \lambda (dD_s + (1 - a_s) \delta_s ds)$$

is a martingale since

$$dV_t = -re^{-r}W_t dt + e^{-r}dW_t + e^{-r} \lambda (dD_t + (1 - a_t) \delta_t dt) = \lambda (1 + \nu/r)(dX_t - a_t \delta_t dt).$$

Therefore, the agent’s expected payoff is

$$E \left[ \int_0^\infty e^{-r} \lambda (dD_s + (1 - a_s) \delta_s ds) \right] \leq E[V_\infty] = V_0 = W_0,$$

with equality if the transversality condition holds. Thus, $W_t$ reflects the agent’s future expected payoff, and it is optimal for the agent to follow the prescribed strategy.

If the agent follows the prescribed strategy, then $W_t$ follows

$$dW_t = rW_t dt - dC_t + \lambda (1 + \nu/r) (dX_t - \delta_t dt)$$

with $dC_t = rW_t dt$ when $W_t = W^d(\delta)$, and $dC_t = 0$ when $W_t < W^d(\delta)$, as in the optimal contract. Moreover, liquidation occurs when the firm’s cash balance drops to the level $M = R_0/\lambda$, and the agent’s value drops to $R(\delta)$, as in the optimal contract. We conclude that the agent’s recommended strategy implements the optimal contract. QED
8. References.


Praveen Kumar, Bong-Soo Lee: Discrete dividend policy with permanent earnings Financial Management, Autumn, 2001


