Trading and valuing toxic assets∗

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Abstract

Fair value accounting forces institutions to revalue their inventory whenever a new trans-
action price is observed. An institution facing a balance sheet constraint can have incentives
to suspend trading in opaque over-the-counter markets to avoid marking-to-market. This
way the asset’s book valuation can be kept artificially high, thereby relaxing the institu-
tion’s balance sheet constraint. But, the institution loses direct control of its asset holdings,
leading to possible excessive risk exposure. An institution optimally balances this trade-off
when choosing the point beyond which it suspends trading. Outside investors, who do not
know at what price the asset would trade, reduce their valuation the longer the asset has
not traded. Their expected discount from book value is convex in time since last trade and
robust to parameter misspecifications.

Keywords: Mark-to-market; Mark-to-model; Level 3 assets; Balance sheet constraints;
Toxic Assets; Optimal Stopping

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Balance sheets matter, especially in times of financial crisis when credit channels tighten. The evaluation of illiquid, or so-called Level 3 assets plays an increasingly important role on banks' balance sheets. Level 3 assets include CDOs, ABSs and other – sometimes bespoke – structured credit products. Such hard-to-value assets made up nearly $600 billion of the balance sheets of the 8 largest US banks as of April 2008, and have been steadily increasing since as more assets have become toxic, i.e. non-traded, over the last year. More recently, many mortgage products became Level 3 assets as the ABX index ceased to serve as a valuation basis.\(^1\) Anecdotal evidence suggests that some assets are even actively kept off markets to obstruct price discovery.\(^2\) Given the accounting flexibility that comes with the Level 3 category, institutions continue to list these assets at inflated values on their books. To outside investors – or the government – involved in takeovers or recapitalization of such firms, the biggest challenge is to ascertain what discount from book values should apply to such non-traded assets. This requires an understanding of when banks suspend trading of these assets.

I provide a model that endogenously derives at what point institutions suspend trading of certain assets, and how to estimate their market value once they have become toxic, i.e. non-traded. Under fair value accounting, book valuations of securities have to be updated when new transaction prices are observed. Over-the-counter markets can be so opaque that no continuously observable prices exist. Thus, an institution active in such markets has to take the accounting impact of its own trading decisions into consideration. In a market with a time-varying liquidity discount, an institution facing a possible balance sheet constraint suspends trading once this discount grows large enough, thereby obstructing further price discovery. As book value ceases to reflect current market prices, the institution’s balance sheet constraint is relaxed. The asset holdings, however, become fixed, leading to potentially excessive risk exposure. The institution optimally balances the benefit of a relaxed balance sheet constraint with the cost of possible excessive exposure to determine when to stop trading. With these derived no-trading bounds, an outside investor – who cannot observe the current liquidity discount – can derive an expected price for the non-traded asset. The expected value of the asset is decreasing and convex in time since last trade, and the discount from its book value is robust to parameter misspecification.

The model is set in a continuous-time stochastic market in which a risk-averse institution

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\(^1\)The ABX index, an index for asset backed securities, does not exist for its theoretically most recent vintage. The old vintages of the ABX have ceased to serve as a valuation basis for many OTC mortgage products (except for the specific names in the index). The website of Markit, the company that owns the Markit ABX.it index, states the following: 30 September 2008 - Per majority dealer vote, the roll date for the Markit ABX.it 05-2 index has been postponed due to the current market conditions. Markit will announce the new launch date in due course. [http://www.markit.com/information/products/category/indices/abx.html](http://www.markit.com/information/products/category/indices/abx.html)

\(^2\)As Norris (2008) writes in the New York Times: "Did you ever hear of a broker who would not agree to earn a commission? Even if getting the money required absolutely no work at all? Apparently, some brokers think such a move could be wise. It’s not that they don’t like income, but they may fear that letting some securities trade at low prices could force them to report even larger losses than they are already posting."
decides to allocate its equity between a risk-free bond and a risky asset. Three assumptions drive the model. First, the institution is subject to a balance sheet constraint, which takes the form of a simple leverage constraint based on book values. Second, the risky asset has a time-varying liquidity discount from fundamental value. This discount gives rise to a shadow price, i.e. the price at which the asset can be bought and sold in the market under current conditions. The shadow price only becomes the realized market price when a trade occurs. Furthermore, in an opaque market the shadow price – as part of the underlying market conditions – is only observed by inside investors. Third, when no continuously observable asset prices or pricing inputs exists, mark-to-market accounting is supplanted by mark-to-model accounting. In this model, an institution is allowed to value the asset at its last reported transaction price for mark-to-model accounting. Furthermore, an institution is able to ignore all but self-generated transaction prices in opaque markets for accounting purposes.

To exploit the time-varying nature of the shadow-price, the optimal strategy absent any constraint is to increase leverage in line with shadow price deviations from fundamental value. A constraint on leverage prevents such a strategy from being implementable. If the market in question is widely traded, the institution cannot obstruct price discovery on its own, and has to comply with the leverage constraint at each point in time. If the market is opaque, however, it is possible for the institution to obstruct price discovery by ceasing to trade. This introduces a trade-off: by continuing to trade, the balance sheet reflects market prices and the constraint applies, but exposure remains fully controlled. By suspending trade, the institution freezes its balance sheet value, thereby relaxing the constraint, but the now fixed asset position can lead to excessive risk exposure.

The resulting problem faced by the institution is complex, as it involves solving a combined optimal asset allocation and optimal stopping problem under a constraint. I use techniques from stochastic dynamic programming and from the theory of linear differential equations to obtain analytic solutions in integral form. These solutions can then be easily evaluated by numerical integration.

Markets that offer the accounting option to conceal current shadow prices from the books will see an institution engaging in such behavior only when the shadow price has fallen far enough. Up until this point the institution trades continuously to adjust its exposure. Once the institution stops trading, further losses on its position are concealed as the balance sheet is frozen. Yet actual leverage – based on the shadow price – continues to drift from reported leverage, thereby allowing a violation of the leverage constraint. There will be a shadow price at which the actual exposure has become too large for the institution’s risk bearing capacity. At this point, it will voluntarily deleverage and reveal all previously concealed losses. If the shadow price reverts before hitting this point of reckoning, the institution will restart trading, having
Figure 1: **Illustrative price path:** The solid horizontal line depicts the *point of concealment* $\hat{x}$ and the dashed horizontal line depicts the *point of reckoning* $\tilde{x}$. As the shadow price is describing a temporarily depressed price, it is bounded between 0 and 1. The shaded area depicts the paths that are concealed from the financial reporting of the institution.

Successfully avoided any constraint induced position adjustments.

This trading behavior will generate a specific valuation profile on the institution's books. When the shadow price is high, the institution will adjust its exposure continuously, and the institution's balance sheet will accurately reflect the asset's market value. Figure 1 illustrates: the paths above the solid horizontal line have a continuously updated balance sheet. Once the shadow price drops low enough, however, trading is suspended and balance sheet valuations become stale. In figure 1, this would correspond to the shaded area below the solid horizontal line. Shadow price innovations on this region are not reflected on the balance sheet, and the valuation stays constant at the value of the solid horizontal line. Near year 8, where the shadow price path hits the dashed horizontal line, we see the voluntary deleveraging due to excessive exposure. As the institution deleverages, the current shadow price is realized, leading to a large downward jump in the book valuation from the solid to the dashed horizontal line. Merrill Lynch faced such a situation in the summer of 2008, when it decided to drastically deleverage out of mortgage related assets. This sale was done at prices significantly below reported valuations, thus leading to a large jump in book value.

A multiple firm extension builds on the same logic, but additionally there is strategic interaction via realized prices. A bank's position defines its incentives to trade or not to trade. Thus, when positions lie closely together because capital constraints are approximately the same, incentives are aligned and a no-trading outcome as in the single bank case can arise. However, when constraints are too far apart, the most constrained firm will not find it optimal to stop trading on the most constrained bank's terms and will trade continuously throughout, removing the no-trade outcome.
Knowledge of the optimal no-trading region will allow an outside investor – who cannot observe the shadow price – to perform a valuation of the company’s non-traded asset. The model shows that the value of such an asset depends on how much time has passed since it was last traded and on its underlying drift and volatility parameters. I calibrate the model and derive the outsider’s valuation by simulation. Surprisingly, conditioning on the no-trading region reverses the upward drift of the shadow price: the expected price is decreasing in time since the last trade. The intuition is that the conditioning eliminates those paths that have the strongest mean-reversion – the ones that revert back to the trading region in the interim. Additionally, the expected discount from the last reported transaction price is robust to misspecification of the mean-reversion parameter $\kappa$, as the trading boundaries optimally adjust to different parameter values.

The technical foundations of my model build on the literature in the area of risky arbitrage. Liu and Longstaff (2004) present a model in which an institution faces a fixed horizon of time $T$ and a Brownian Bridge as the price process. They show that even in this case of a pure arbitrage opportunity (the asset mean-reverts a.s. at the horizon) the institution can nevertheless face risk in the presence of margin constraints. Since the authors do not allow for possible accounting manipulation, the trade-off between no trading and stale valuations on the one hand and continuous trading on the other hand cannot arise in their framework. In contrast to Liu and Longstaff (2004), this paper also presents a closed form solution to the value function in the constrained case with an exponential OU process. Other models allowing for portfolio constraints are Grossman and Vila (1992), Cvitanic and Karatzas (1992), Pavlova and Rigobon (forthcoming) and Basak and Croitoru (2000).

Related papers focusing on the strategic interaction originating from balance sheet constraints are Brunnermeier and Pedersen (2005) and Attari, Mello, and Ruckes (2005). Here, competitors try to exploit the balance sheet constraint of an institution via market-based price manipulation. This effect propagates through an assumed temporary price-impact of trades.

There are also several related models that consider an unconstrained asset allocation problem with mean-reverting asset prices. Jurek and Yang (2007) extend the notion of risky arbitrage to include timing arbitrage. Using an OU process for the asset price with a fixed horizon, they are able to solve the asset allocation problem of an unconstrained institution for general CRRA utility functions. Similarly, Kim and Omberg (1996) solve a model with a mean-reverting Sharpe-ratio and a fixed horizon for the HARA class of utility functions. Xiong (2001) presents an equilibrium model involving a mean-reverting cash-flow process.

A related paper with respect to the fair value accounting component of the model is Plantin, Sapra, and Shin (forthcoming). The paper examines the systemic effects of marking-to-market in a static model that utilizes a game theoretic framework. The authors show that mark-to-market
accounting can lead to a destabilization of the financial system in that it enhances feedback effects.

The paper is structured as follows. Section 1 sets up the model, and discusses the balance sheet constraint and accounting assumptions. Section 2 establishes preliminary results for the case of continuous balance sheet updating with and without constraints. Section 3 presents the main results of the paper, solving the model under the option to keep valuations stale. Section 4 provides an extension covering the interaction between multiple banks. Section 5 applies the previously derived optimal no-trading bounds to the valuation of toxic, i.e. non-traded, assets. Section 6 concludes.

1 Model Setup

The model is set in continuous-time with \( t \in [0, \infty) \) on a suitable defined probability space \((\Omega, \mathbb{P}, \mathcal{F})\). There is one type of agent, a financial institution such as a bank, or a proprietary trading desk within a bank. My goal is to characterize the dynamic impact of accounting rules and balance sheet constraints on the institution’s trading decisions and consequently on the path of reported valuations. The optimal no-trading boundaries will in turn allow the derivation of the expected price of non-traded assets to outside investors with no access to market prices. Further, I interpret the term toxic to refer to an asset being non-traded. For tractability, I set the interest rate to zero, i.e. \( r = 0 \).

Shadow price process. Assets in thinly traded markets, such as certain OTC markets, can be subject to significant deviations of realized prices from fundamentals. As Brunnermeier and Pedersen (forthcoming) argue, such a temporary price depression can be caused by market participants being subject to funding liquidity constraints or the possibility of such in the future. Also, temporary price depression can arise naturally in OTC markets from imbalances in buy and sell orders. A paper that addresses these issues in a matching framework is Duffie, Garlenau, and Pedersen (forthcoming). Furthermore, Mitchell, Pedersen, and Pulvino (forthcoming) find that even relatively large imbalances in price are not immediately restored because capital is slow moving: certain levels of expertise are required in such thin markets that cannot be acquired instantaneously.

For simplicity, there is only one risky asset the institution can invest in, simply termed the asset. The asset pays no dividends, but its price \( P_t \) can be temporarily depressed. \( P_t \) should be understood as the price at which the asset can be bought or sold at time \( t \) as the outcome of a matching process. Once a trade occurs at \( t \), the shadow price becomes the realized transaction price. The institution that is active in the market observes the shadow price even when no
transaction occurs, but outside investors – or the government – cannot observe the current shadow price. Let \( x \) be the log shadow price process, so that

\[ P_t = \exp(x_t) \]  \hspace{1cm} (1)

I model the shadow price to be mean-reverting by assuming that the log shadow price \( x_t \) follows an OU process with mean zero. The initial shadow price is \( P_0 = 1 \), i.e. \( x_0 = 0 \). To model a temporary price depression, I reflect the process \( x \) downward at 0. Note that such a reflection bounds the shadow price between 0 and 1 and shifts the mean of \( x \) to below zero. The process remains attracted to fundamental value \( x_t = 0 \).\(^3\) Let

\[ dx = -\kappa x dt + \sigma dZ - dK \]  \hspace{1cm} (2)

where \( \kappa > 0 \) is the mean reversion parameter, \( \sigma > 0 \) is the constant instantaneous volatility parameter, \( dZ \) is a standard Brownian motion and \( dK \) is the singular process governing the reflection.\(^4\)

The reflection should be understood as approximating a process near 0 that has \( x = 0 \) as an inaccessible boundary (e.g. a Feller process). Alternatively, we could impose a change in the process from an OU process to a GBM in an \( \varepsilon \)-distance from 0, leading to much more technical detail. As the focus of the model, however, is on behaviour away from the boundary we pick the most tractable setup around 0. For additional tractability, we make two simplifying assumptions: (1) the agent cannot short, and thus cannot exploit the behavior around \( x = 0 \), and (2) we ignore the local “stop-loss” wealth loss that might be generated by the trading strategy (i.e. we ignore the possible local time generated by the trading strategy as in Carr and Jarrow (1990)).\(^5\)

By Ito’s Lemma, the shadow price will follow the process

\[ \frac{dP}{P} = \left(-\kappa x + \frac{1}{2} \sigma^2\right) dt + \sigma dZ - dK \]  \hspace{1cm} (3)

As mentioned above there is another group of possibly funding liquidity constrained investors

\(^3\)As the mean is shifted below zero, referring to the log price \( x \) attraction to 0 as ‘mean-reversion’ is strictly speaking incorrect. I will, however, continue to use the term ‘mean-reversion’, with abuse of terminology, as referring to \( x \)’s attraction to 0.

\(^4\)This specification abstracts from price impact for tractability. The discontinuous impact on the books brought about by marking-to-market when realizing a possibly much lower price is present even for very small, \( \varepsilon \)-volume trades. Any additional price impact would be in some way proportional to a power of trading volume \( \varepsilon \). Such price impact would result in lower ‘volume’ trades when trades occur, but would only affect the decision to trade or not to trade to a second order.

\(^5\)Alternatively, but with no additional insight, we could introduce temporary price overshooting by letting OU range from \((-\infty, \infty)\), removing the need for additional assumptions to handle \( x = 0 \). This comes at the cost of doubling the number of parameters to be estimated without affecting the main insight of the model.
ready to buy and sell at the prevailing shadow price. In this sense, the market is frictionless. The possible trading impact of the institution is caused by the balance sheet link.

For the rest of the paper, we calibrate the mean-reversion and volatility parameters to the values $\kappa = 0.8$ and $\sigma = 0.4$. The calibrated parameters yield an average shadow price of 80 cent on the dollar. The expected time for the average mispricing to disappear completely is 5 month.

By excluding asymmetric information considerations, we are able to isolate the mechanical effects fair value accounting and balance sheet constraints have on trading behavior. In the current crisis there is evidence of strong balance sheet driven effects influencing the trading behavior of institutions, as for example presented by Adrian and Shin (2008).

The model as presented is based on temporary price depressions. In the current crisis there are also permanent drops in value. The results would still hold under some additional fundamental uncertainty, i.e. there would still be some regions of no-trading, as long as there is some element of mean-reversion in the price in our current setup. The closed form solution, however, would be lost. From a modeling viewpoint, the mean-reversion property here is not a key assumption for the mechanism. What is needed in the model is that the agent has some region on which his optimal unconstrained strategy is unattainable because of the balance sheet constraints. Hiding market valuations by marking-to-model can allow the agent to better approximate parts of the optimal unconstrained strategy in this region. The mean-reverting assumption results in one specification that best fits our story.

**Wealth process.** Financial institutions largely finance themselves through the standard instruments of debt and equity. At a bank, management’s task is to maximize the value of existing equity. Similarly, a proprietary trading desk within a bank will have incentive systems in place that will drive employees to maximize trading profits and thus the desk’s equity.

Let $W_t$ denote the wealth or equity of the institution. The balance sheet in this simple world is made up of debt $D_t$ (negative $D_t$ denotes lending) with constant price 1 and $N_t$ units of the risky asset of price $P_t$ each (negative $N_t$ denotes shorting).

\[
\begin{array}{c|c}
\text{Assets} & \text{Liabilities} \\
\hline
N_t \times P_t & D_t \text{ Debt} \\
\text{Value of Assets} & W_t \text{ Wealth / Equity} \\
\end{array}
\]

Naturally, $W_t$ is restricted to be non-negative. Without loss of generality, we will be able to ignore this constraint due to our assumption of logarithmic utility.

The bank’s choice variable is $\phi = \frac{NP}{W}$, its leverage or exposure. Because we assume logarithmic utility, it will be useful to derive the log wealth dynamics. Applying Ito’s Lemma to
\[ \log (W_t), \text{imposing self-financing and recalling that } r = 0, \text{ we have} \]

\[
d \log W = \frac{1}{W} N dP - \frac{1}{2} \frac{1}{W^2} \langle N dP \rangle^2
\]

\[
= \left( \left( -\kappa x + \frac{1}{2} \sigma^2 \right) \phi - \frac{1}{2} \sigma^2 \phi^2 \right) dt + \sigma \phi dZ - \phi dK
\]

With \( \phi \) as a choice variable, the drift of \( \log W \), \( \mu_{\log W} \), is independent of the level of \( W_t \). Note further that \( W_t \) by itself is not Markovian, but that the two dimensional process \((W_t, x_t)\) is. The bank is endowed with a strictly positive and finite amount of initial wealth \( W_0 \).

For integrability, I impose that \( \phi \) has to be square integrable, i.e. \( \phi \in L^2 \). I additionally rule out any recapitalization of the bank as it would allow adjustments of \( \phi \) without trading.

**Preferences & Time horizons.** We will abstract from management fees or any other stream of intertemporal utility and assume that the institution maximizes its utility from final wealth at a random time \( \tau_{\rho} \). Thus, the institution has no intermediate consumption. It liquidates all its holdings at time \( \tau_{\rho} \) and realizes a utility \( U(W_{\tau_{\rho}}) = \log (W_{\tau_{\rho}}) \).

The stopping time \( \tau_{\rho} \) is an exponentially distributed random variable with intensity \( \rho \) that is independent of \( Z \). The value function can thus be written as

\[
V_t(W, x) = \sup_{\phi} \mathbb{E}_t [\log (W_{\tau_{\rho}})] = \sup_{\phi} \mathbb{E}_t \left[ \int_{t}^{\infty} \rho e^{-\rho(s-t)} \log (W_s) \, ds \right]
\]

where \( \mathbb{E}_t [\cdot] \) denotes the expectation operator w.r.t. to the filtration \( \mathcal{F}_t \). Note that due to log utility, wealth always stays positive, making debt risk-free.

If the institution had a long enough horizon, it could exploit the attraction of \( P_t \) to fundamental value 1 to achieve infinite wealth. The random but almost surely finite liquidation time \( \tau_{\rho} \) exposes the institution to the risk of having to liquidate asset positions on paths of large deviations and realize potentially large losses. In terms of the benchmark calibration, we will assume \( \rho = 1/8 \), yielding an average liquidation time of 8 years.

The choice of preference structure, although simplistic, allows us to capture the essential parts of market participants' behavior. First, decision making units at financial institutions have finite time horizons, though the exact horizon remains uncertain. Alternatively, the horizon can be understood as an extreme liquidity event that forces the realization of wealth, and thus prices, onto the bank. Second, investment banks display effective risk-aversion. To focus on the economic aspect of balance sheet constraints, we use a logarithmic utility definition, as it allows

\footnote{By the continuity property of analytical solutions, our results also hold for constant relative risk aversion utility functions close to logarithmic utility.}
to abstract from any possible intertemporal hedging demands.

**Leverage constraint.** Financial institutions are constrained in their asset allocation decisions: financial market regulators require certain amounts of minimum capital, prime brokers demand protection via margins and debt contracts often come with attached covenants. Any such constraint will curtail the institution’s ability to exploit large deviations of price from its long-run equilibrium value by restricting the maximum position size. To make a statement about the impact of fair value accounting on an institution’s portfolio decisions, we will focus on constraints that operate on the reported balance sheet of the institution.

Regulatory capital requirements force a bank to have sufficient capital base for their asset positions. Such a capital requirement can be found in the Basel accords. We assume here that the regulator uses the reported balance sheet data for such capital requirements. A minimum capital requirement then gives a static leverage constraint in our model.

Additionally, debt often comes with attached covenants to limit the risk-taking ability of the firm. Such debt covenants fit into our story if they can be (a) summarized as a static leverage constraint and (b) are only enforceable in courts on the reported statements of the firm (i.e. the accounting statements).

If we follow the interpretation that the decision making unit is a trading desk within an institution, the object of focus will shift to in-house capital allocation. As in the above case, the desk’s balance sheet will play an important role with regard to sustained funding. We assume that internal capital allocation and risk-management constraints can be summarized by a static leverage constraint on the reported internal balance sheet.

In terms of our model, the balance sheet constraint takes the form of a leverage constraint

$$\phi \in [\underline{\phi}, \bar{\phi}]$$

(6)

where we assume $\underline{\phi} = 0$ (a no-shorting constraint) and $\bar{\phi} \geq 1$: the institution is allowed to invest all of its equity in the risky asset ($\phi = 1$) or in the bond ($\phi = 0$). In this model only $\bar{\phi}$ will matter, and we will thus refer to $\bar{\phi}$ in the following simply as the constraint.

**Accounting.** Recent shifts away from historical cost accounting to fair value accounting have led to significant changes in how financial assets are valued. For accounting purposes, there are three asset categories, as specified in Financial Accounting Standards Board ("FASB") statement 157 that implemented fair value accounting: Level 1, Level 2 and Level 3.\(^7\)

It is important to note the position of the SEC regarding the primacy of a market price in accounting valuations. A recent report by the Center for Audit Quality (2008) argues that it is

\(^7\)FASB 157 became effective November 15, 2007.
"important to distinguish between an imbalance between supply and demand (e.g., fewer buyers than sellers, thereby forcing prices down) and a "forced" or "distressed" transaction". Although FASB 157 allows for these valuation exceptions, the interpretation of this rule by the regulator is strict. Applying a forward looking equilibrium view of valuations that ignores current illiquidity or imbalances in supply and demand is not permissible, as "GAAP defines fair value as the amount at which an asset could be bought or sold in a current transaction". However, with FASB 159 having been passed in February 2009, there has been a significant expansion of when it is allowed to use marking-to-model in illiquid markets.

To model marking-to-market and marking-to-model in a tractable way, I make the following simplifying assumptions: (1) The institution has the option to value the asset on its books at its last observable transaction price. In other words, in a market with no observable prices, trading is intimately linked to price realization. (2) Once an institution trades at a price that is below its accounting valuation, it is forced to adjust its position immediately if in violation of the leverage constraint. For tractability, if in violation, the institution is forced by the regulator to liquidate all holdings of the asset at the current shadow price and exit the market.

Allowing the institution to mark to the last observable transaction price permits us to remain agnostic about the model used by the institution to value the asset. This affords us a certain degree of robustness while still maintaining the essential link between an institution’s own trading decision and the value of its inventory. In real life, internal auditors are informed at an almost instantaneous basis of new transaction prices that are generated by the bank itself. Furthermore, the accounting balance sheet of the trading desk is reported daily to the regulator.

Let us briefly discuss the three asset classes.

Level 1 assets. Level 1 assets are liquid assets with publicly quoted prices, such as exchange traded assets. The unadjusted quoted prices have to be used for accounting purposes if the company has access to the market in question. Thus, the price of these assets is close to unambiguous and the institution has virtually no discretion in valuing its books. Examples of such assets would be common stock traded on the NYSE, such as a GM share.

Level 2 assets. Although Level 2 assets are not exchange traded, the valuation of these assets is based on a model that requires the input of market observables. Such observables might be quoted prices of similar assets, interest rates, implied volatility or a related index. Level 2 assets, although market-to-model, are thus subject to continually updated inputs. Examples of Level 2 assets are simple derivatives such as a plain vanilla option on a common stock or some OTC derivatives with an observable index.

As Level 2 assets have to be marked-to-model based on observable inputs, there are only few degrees of freedom available to the institution in how to influence the valuation. This motivates
the decision to treat Level 2 assets as having continuously updated valuations in this model.\(^8\)

**Level 3 assets.** According to FASB 157, Level 3 assets are such assets that have to be marked-to-model based on unobservable inputs. For the category Level 3 to apply, there have to be neither quoted prices of the asset itself nor any observable, i.e. Level 2, inputs available to the institution. This means that there are no actively traded similar or identical assets, nor are there any observable market inputs such as relevant indices or underlying assets. It is clear from the description that Level 3 assets are primarily found in OTC markets with high levels of opaqueness.

To model the valuation of Level 3 assets in a tractable way, we will make a third assumption: (3) Unless stated otherwise, the institution operates in an opaque enough market to be able to ignore other institution’s transaction prices of the same or similar assets for its own accounting treatment. This is a reasonable assumption in the OTC markets we are trying to model, and will allow to abstract from possible strategic (or predatory) motives for trading. Financial accountants and auditors, even internal ones, do not have the expertise to be able to observe the shadow price process, nor do they have access to transaction prices of other institutions because of the decentralized nature of the market.\(^9\) We relax the non-observability assumption in section 4.

## 2 Continuously updated valuations: Level 1 and 2 assets

In this section, we will look at the case of Level 1 and Level 2 assets for which accounting rules result in continuously updated valuation. We will solve for the optimal trading rules and value functions of a single bank with and without a leverage constraint that will provide upper and lower limits respectively for the Level 3 case.

Denote the value function of the overall problem by \(V(W,x)\). Using standard dynamic programming techniques, we can derive the Hamilton-Jacobi-Bellman ("HJB") equation that

\[^8\]There is a possible exception in that accounts can be declared "hold to maturity". This option is often utilized by insurance companies, as many of their claims are not retracted. Declaring an account as "hold to maturity", however, restricts the ability to retrade in the future: trading non-trivial amounts of the asset will force the institution to re-declare the account as a trading or assets available for sale account, with the option to switch back to "hold to maturity" only after a fixed period of time.

\[^9\]Current practice has internal auditors gathering quotes from other institutions if those provide them voluntarily. But even if such quotes are available, it is important to note that these are not trade quotes – they come from the other institution’s risk management units, and not from the actual trading desks. Clearly, if there is no threat of trade on these quotes, and if institutions are in similar situations, there is little incentive to report the true actual quotes.
results in a partial differential equation ("PDE") for \( V(W, x) \) on \( x \in (-\infty, 0) \)

\[
\rho V = \max_{\phi \in A} \left[ -\kappa x V_x + \phi \left( -\kappa x + \frac{1}{2} \sigma^2 \right) W V_w + \frac{1}{2} \sigma^2 V_{xx} + \frac{1}{2} V_{ww} W^2 \sigma^2 \phi^2 + \sigma^2 \phi W V_{xw} + \rho \ln(W) \right]
\]

where \( A \) is the restriction on \( \phi \), such as for example \( A = [0, \bar{\phi}] \). Ignoring the “stop-loss” wealth loss, the boundary condition is \( V_x(W, 0) = -\phi(0) \).

As we have log utility, we conjecture an additive separable form of the utility function in \( W \) and \( x \):

\[
V(W, x) = \log W + g(x)
\]

Note that \( g(x) = 0 \) at the horizon time \( \tau \), as the institution liquidates its holdings for a wealth of \( W_\tau \) and realizes utility \( \log W_\tau \). In this sense, \( g(x) \) is the option value of having access to the market for the risky asset, and we must have \( g(x) \geq 0 \). This option is non-traded and thus influenced by the possible constraints the institution might be facing.

The separable form of the value function allows us to substitute out the terms involving \( W \) and derive an equation involving only \( x \). The PDE reduces to an ordinary differential equation ("ODE") for the function \( g(x) \). We will refer to \( g(x) \) as the value function if no confusion can arise.

\[
\frac{1}{2} \sigma^2 g''(x) - \kappa x g'(x) - \rho g(x) + \max_{\phi \in A} \left\{ -\kappa x + \frac{1}{2} \sigma^2 \phi - \frac{1}{2} \sigma^2 \phi^2 \right\} = 0
\]

Maximizing the value function w.r.t. \( \phi \) thus reduces to maximizing the drift of \( \log W, \mu_{\log W} \).

**Continuously updated valuations without leverage constraints.** When the agent is unconstrained, i.e. \( A = (0, \infty) \), the first order condition results in a linear investment policy:

\[
\phi^* = \left\{ -\frac{\kappa}{\sigma^2} x + \frac{1}{2} \right\} 1_{\{x \neq 0\}}
\]

This is a classic result of the Merton model: a logarithmic agent’s portfolio position in the assets is simply the asset price’s expected growth rate (from equation (3) this is \(-\kappa x + \sigma^2/2\)) normalized by the asset’s instantaneous volatility (from equation (3) \( \sigma^2 \)). Also, because of our reflection specification, the agent trades down to 0 to avoid any exposure at \( x = 0 \).

Observe that the boundary condition becomes \( g'(0) = -\phi^*(0) = 0 \). When \( x \) decreases the asset becomes more attractive as its upward drift increases. The institution optimally exploits this
raised drift by increasing its leverage.\textsuperscript{10} As the institution is able to trade continuously, exposure never increases fast enough to possibly push the institution into negative wealth territory.

We are now ready to establish our first result. All proofs can be found in the appendix.

**Proposition 1** The value function of the unconstrained institution is 
\[ V(W,x) = \log W + g(x) \]
with
\[ g(x) = Ax^2 + Bx + C + c_U U(x) \]  \hspace{1cm} (11)
where the coefficients \( A, B, C, c_U \) are given in the appendix. \( U(x) \) is a Kummer function as defined in the appendix. \( g(x) \) is decreasing in \( x \).

Figure 2 shows the value function \( g \) and the corresponding optimal exposure \( \phi^\ast \) for our benchmark case of \( \kappa = .8, \sigma = .4 \) and \( \rho = 1/8 \). \( U(x) \) vanishes rapidly as \( x \) decreases, so the value function is approximately quadratic. The unconstrained value function will serve as an overall upper limit for any constrained value functions. Even when \( x = 0 \), there is a positive option value \( g(0) > 0 \) as the price process can diverge in the future. With the analytical form in hand, we are now in a position to describe derivatives of the unconstrained value function w.r.t. the parameters around \( x = 0 \).

**Corollary 1** The unconstrained value function \( V(W,x) = \log W + g(x) \) has the following derivatives in an open interval around its reflection point \( x = 0 \): The value function is
(i) increasing in the mean-reversion parameter \( \kappa \) : \( \frac{\partial g(0)}{\partial \kappa} > 0 \)
(ii) increasing in the instantaneous volatility parameter \( \sigma \) : \( \frac{\partial g(0)}{\partial \sigma} > 0 \)
(iii) decreasing in the liquidation parameter \( \rho \) : \( \frac{\partial g(0)}{\partial \rho} < 0 \)

An increase in \( \kappa \) can be understood as a decrease in the average time it takes a given price depression to disappear. As this process is what generates value to the agent, a quicker reversion time generates more value. Similarly, a higher \( \sigma \) leads to more price depressions to occur, also resulting in a higher value function. Finally, the smaller \( \rho \), the longer the time horizon of the agent and the more time to exploit the price dynamics.

**Continuously updated valuations with leverage constraints.** Let us now impose a leverage constraint as discussed in Section 1. Once the constraint is binding, the institution’s ability to exploit deviations from fundamental value is curtailed. Novel results derive from the analytic\textsuperscript{10}The actual position adjustment in \( N \) is more complex. Consider keeping \( N \) constant. The local derivative of \( \phi \) (as shown later in equation (15)) for keeping \( N \) fixed w.r.t. \( x \) is \( -\phi(\phi - 1) \). We can compare this to \( \frac{d\phi^\ast}{dx} = -\frac{\kappa}{\sigma^2} \), the slope of the optimal policy \( \phi^\ast \). When \( -\phi(\phi - 1) < -\frac{\kappa}{\sigma^2} \), the institution is selling units as \( x \) decreases. However, when \( -\phi(\phi - 1) > -\frac{\kappa}{\sigma^2} \), the institution increases its holdings of the assets as \( x \) decreases. In both cases, the institution increases its leverage \( \phi^\ast \).
Figure 2: Value functions \( g(x) \), \( \tilde{g}(x) \), \( \tilde{\tilde{g}}(x) \) and corresponding leverage for parameter values \( \tilde{\phi} = 1.6, \kappa = .8, \sigma = .4, \rho = 1/8 \) as a function of the log shadow price \( x \). The optimal no-trading boundaries are marked by the two horizontal lines \( \hat{x} \) (left line) and \( \hat{\tilde{x}} \) (right line) in both panels. The top panel depicts the value functions for the unconstrained \( g(x) \) (black), constrained w marking-to-market \( \tilde{g}(x) \) (red), and constrained w stale valuations/marketing-to-model \( \tilde{\tilde{g}}(x) \) (blue) case. The bottom panel depicts the corresponding optimal leverage.
solution that allows to quantify the magnitude of utility loss brought about by the leverage constraint.

As the chosen preference structure essentially makes the institution myopic with respect to the constraint, deriving the optimal strategy is straightforward. With $A = (0, \bar{\phi})$ we have

$$\bar{\phi}^* (x) = \begin{cases} \phi^* (x) & \text{for } x \in [\bar{x}, 0] \\ \bar{\phi} & \text{for } x \in (-\infty, \bar{x}] \end{cases}$$

(12)

The institution follows the unconstrained optimal policy $\phi^* (x)$ until the leverage constraint binds at point $\bar{x} = -\frac{\sigma^2}{\kappa} \left( \bar{\phi} - \frac{1}{2} \right)$. Beyond this point, it is forced to keep leverage constant at its highest permissible level $\bar{\phi}$. The optimal strategy is myopic – the possibility of the constraint being binding in the future does not influence the investment decision today.

Once the institution is in the constrained region $x \leq \bar{x}$, wealth is simply driven by the constant leverage $\bar{\phi}$. On the unconstrained region, wealth remains driven by the unconstrained optimal leverage $\phi^*$. This leads to two different ODEs, one for each region. Pasting the resulting solutions together at the point $\bar{x}$ and imposing the relevant boundary conditions, we derive the value function.

Let a superscript $c$, i.e. $\bar{y}^c$, denote parameters referring to the case of the constrained institution facing continuously updated prices. Furthermore, let an upper bar $\bar{\cdot}$, i.e. $\bar{y}$, denote parameters and functions corresponding to the region that has continuous trading while the constraint is binding.

**Proposition 2** The value function for the constrained institution without the option to mark-to-model is $\bar{V} (W, x) = \log W + \bar{\bar{g}} (x)$ with

$$\bar{g} (x) = \begin{cases} Ax^2 + Bx + C + c_M M (x) + c_U U (x) & \text{for } x \in [\bar{x}, 0] \\ \bar{B}x + \bar{C} + \bar{c}_M M (x) + \bar{c}_U U (x) & \text{for } x \in (-\infty, \bar{x}] \end{cases}$$

(13)

where the coefficients $A, B, C$ are the same as in the unconstrained case. The Kummer functions $M (x), U (x)$ and the parameters $c_U, \bar{B}, \bar{C}, c_M, \bar{c}_M$ are given in the appendix.

Figure 2 shows the value function $\bar{g}$ and the corresponding optimal exposure $\bar{\phi}^*$. We can now examine the magnitude of the value loss in this region brought about by the leverage constraint. Again, $U (x)$ vanishes rapidly as $x$ decreases, so the value function essentially becomes linear on the constrained region $x \leq \bar{x}$. The institution therefore loses value of an order of magnitude in $x$. Of course, if the constrained region is remote in a probabilistic sense, this will not have much effect on the value function close to $x_0 = 0$.

The behavior of $\bar{g}$ at $x = 0$ w.r.t. the model’s parameters follows our earlier discussion for
behavior of the unconstrained value function $g$.

The coefficient on $U(x)$ on the unconstrained part remains the same as in the case without the constraint, i.e. $c_U^c = c_U$, as it is determined solely by a boundary condition at the reflection point 0 that is independent of $\tilde{\phi}$. The only difference between the unconstrained value function $V$ and the constrained value function $\bar{V}$ on $[\bar{x}, 0]$ is therefore the term $c_M^c M(x)$. Comparing different value functions indexed by $\tilde{\phi}$ on $[\bar{x}, 0]$ thus reduces to comparing the different $c_M$'s.

Noting that $M(0) = 1$, we have by simple optimality of the unconstrained solution that $c_M^c < 0$. Furthermore, by the option character of investing we must have $\bar{g}(x) \geq 0$. Thus, we have $c_M^c \in (-\bar{g}(0), 0)$.

We can now derive a monetary measure of the value loss that arises from the leverage constraint. Consider raising the institution’s wealth by a factor of $\alpha$. What value of $\alpha$ would make the institution indifferent between being unconstrained with initial wealth $W_0$ and facing a leverage constraint $\bar{\phi}$ with raised initial wealth $\alpha W_0$? The indifference point $\alpha$ for $x_0 = 0$ satisfies

$$V(W_0, x_0) = \bar{V}(\alpha W_0, x_0) \iff \alpha = \exp(-c_M^c) > 1$$

For a graphical examination of $\alpha$ the reader is referred to the graphics appendix. For our benchmark parameters, $\alpha$ is monotonically decreasing in $\bar{\phi}$, with a constraint of $\bar{\phi} = 1.6$ requiring 50% more initial wealth.

### 3 Possibly stale valuations: Level 3 assets

This section contains the main proposition of this paper that derives the optimal strategies and value function in the case of Level 3 assets, when the institution’s trading decision can impact its financial reporting. This in turn will allow us to make statements about the estimation of the liquidity discount for non-traded assets later on in section 5.

For the rest of this section, we will use the following terminology:

- We will denote by $\hat{x}$ the so called *concealment point* below which the institution will stop trading.

- Denote by $\bar{x}$ the so called *point of reckoning* at which the institution *voluntarily* reveals the shadow price, is forced to liquidate its position due to a violation of the leverage constraint and realizes its wealth $\log W$.

Due to the assumption of complete liquidation once the bank is revealed to be in violation of the
leverage constraint, the possible optimal concealment strategy can be summarized by the 2-tuple \((\hat{x}, \tilde{x})\). This gives rise to a two-sided optimal stopping problem: one optimal stopping point \(\hat{x}\) and one optimal transitional point \(\tilde{x}\). The optimal stopping point thus comes with an 'outside option' of 0.

Given our assumptions from section 2, the institution holding Level 3 assets faces the following situation: If it trades at time \(t\), the shadow price \(P_t\) is realized and accounting rules force an updated valuation of any inventory still held. If no own trade occurs, however, the bank can keep level 3 assets at their old values on the books.

**Possibly stale valuations without leverage constraints.** The option to keep valuations stale only generates value if it allows a relaxation of the balance sheet constraint. As this constraint is nonexistent for \(\bar{\phi} = \infty\), the institution will find it optimal to continuously trade throughout the domain of \(x\), marking-to-market its balance sheet at each point in time. Thus, the unconstrained solution \(V\) from section 2 is the upper limit to any value function involving accounting manipulation.

**Possibly stale valuations with leverage constraints.** Once the institution faces a constraint on its leverage, the option to keep valuations stale becomes valuable. The intuition for this is simple: with continuous trading and thus continually updated prices, the institution is forced to fulfill the leverage constraint \(\phi_t \leq \bar{\phi}\) a.s.. Consequently, it has to accept suboptimal leverage on the constrained region \((-\infty, \bar{x})\). With marking-to-model, concealing losses via stale book valuations, the institution is able to keep its book value artificially inflated and thereby relax the leverage constraint for some range of \(x\).

Denote by \(\bar{\phi}\) the institution’s actual exposure based on the current shadow price, as opposed to its accounting based exposure \(\phi_t\) based on a possibly stale transaction price. As long as the institution is trading continuously, these two variables will coincide, i.e. \(\bar{\phi} = \phi_t\). When the institution stops trading at a time \(s\), its number of units of the risky asset, \(N_s\), and the amount of debt, \(D_s\), become fixed (recall \(r = 0\)). Let \(\hat{x}\) be an arbitrary concealment point and \(\bar{\phi} \equiv \phi_t(\hat{x})\) \(\leq \phi\) be the arbitrary leverage at this point. Then we can prove the following lemma about the finiteness of the (optimal) point of reckoning for any given finite concealment point.

**Lemma 1** Let \(\bar{\phi} > 1\). If, for any concealment point \(\hat{x}\), the institution follows an investment strategy \(\bar{\phi}^*(x)\) of the form

\[
\bar{\phi}^*(x) = \begin{cases} 
\bar{\phi}^*(x) & \text{for } x \in (\hat{x}, 0] \\
\bar{\phi} & \text{for } x = \hat{x}
\end{cases}
\]

then:

(i) On the concealment or no-trading region \(x \in (\hat{x}, \bar{x})\), the institution’s actual leverage \(\tilde{\phi}\) is only
a function of the distance of the current state $x$ from the concealment point $\hat{x}$:

$$\tilde{\phi}_t (x, \hat{x}) = \frac{\tilde{\phi} e^{(x-\hat{x})}}{1 + \phi \left[ e^{(x-\hat{x})} - 1 \right]} > \hat{\phi}$$

(14)

(ii) The wealth or equity of the institution becomes zero at a finite point $\overline{x}$, the bankruptcy point, defined by

$$\overline{x} = \hat{x} + \log \left(1 - 1/\tilde{\phi}\right) < \hat{x}$$

(iii) The optimal point of reckoning $\tilde{x}$ therefore has to lie in the finite interval

$$\tilde{x} \in (\overline{x}, \hat{x})$$

Proof. (i) By simple substitution, and recalling that $\tilde{\phi} = \frac{N_s P_t}{W_s}$ and $r = 0$, we can derive

$$\tilde{\phi}_t (x, \hat{x}) = \frac{N_s P_t}{W_t} = \frac{N_s P_t}{N_s P_t - D_s} = \frac{N_s P_t}{W_s + N_s (P_t - P_s)} = \frac{\tilde{\phi} P_t}{1 + \phi \left[ e^{(x-\hat{x})} - 1 \right]} > \hat{\phi}$$

(15)

(ii) First, note that the numerator of $\tilde{\phi}$, $N_s P_t$, is everywhere positive and bounded. Thus, $\tilde{\phi}$ has a pole at the point at which the denominator becomes zero, $\overline{x} = \hat{x} + \log \left(1 - 1/\tilde{\phi}\right)$. As $\tilde{\phi} > 1$, we conclude $\overline{x} < \hat{x}$.11 Second, note that the actual exposure $\tilde{\phi}$ becomes negative for $x < \overline{x}$. Thus wealth has to vanish at $\overline{x}$.

(iii) By its definition $\overline{x}$ is finite for any finite $\hat{x}$ and $\tilde{\phi} > 1$. Thus, the set $\{ t : x_t = \overline{x} \}$ has non-zero probability mass. We conclude that the value function $\tilde{V}$ of the institution for any point of reckoning $\tilde{x} \leq \overline{x}$ would result in a value of $-\infty$. This is clearly suboptimal, as any $\tilde{x} > \overline{x}$ yields finite value. We therefore conclude that $\tilde{x} > \overline{x}$. ■

Part (i) of the lemma confirms our previous intuition that stale prices allow the institution to expand its space of possible exposure $\tilde{\phi}$ beyond $[\phi, \tilde{\phi}]$, albeit in a way that is linked to the behaviour of $x$. This expanded investment space can prove valuable if it allows the institution to closer approximate the optimal unconstrained strategy $\phi^*$ for some range of prices.

This closer approximation of $\phi^*$ via $\tilde{\phi}$ comes at the cost of a shorter horizon as part (ii) shows. As $x \rightarrow \overline{x}$, the institution’s actual exposure becomes unbounded, i.e. $\tilde{\phi} \rightarrow \infty$. By using a concealment strategy, the institution thus takes on the possibility of very risky wealth. One path of such actual exposure can be seen in Panel B of figure 2: the solid line depicts actual exposure

11We ruled out $\tilde{\phi} \in [0, 1]$ by assumption. It should be clear to the reader that in this case the institution’s wealth can never become negative as $\tilde{\phi} \in [0, 1]$. It follows that the concealment option is never exercised.

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\( \phi \) if the institution stops trading at a point \( \hat{x} \). Actual exposure \( \phi \) starts increasing rapidly beyond a certain point, increasing the likelihood of negative wealth. Consequently, the institution will liquidate before the exogenous final date \( \tau_\rho \) if the log shadow price reaches a point \( \tilde{x} \), as shown in part (iii) of the lemma.

Once the institution decides to stop trading, only two outcomes are possible: Either the institution liquidates, or the price reverts back to the trading region and trading resumes. On \( (\tilde{x}, \hat{x}) \) there is a time value to waiting, as the underlying log shadow price process \( x \) has a drift that pulls it back towards 0. However, as \( x \) is stochastic, prices can deteriorate further before they rebound. By having the option to liquidate at its discretion, the institution can ensure itself positive wealth. As \( x \) deteriorates, there will be a point \( \bar{x} \) at which the actual exposure of the institution is so large that the time value to waiting becomes zero.

An optimal trading strategy \( (\hat{x}, \tilde{x}) \) will thus result in a specific reported valuation profile. Figure 1 illustrates. The following 2 regions characterize the price path, where \( \bar{P} \equiv P(\tilde{x}) < \hat{P} \):

1. On \( x \in [\hat{x}, 0] \) there is continuous trading in the market and the transaction prices \( P_t \) will be continuously updated. In figure 1, this is the region above the solid horizontal line.

2. When the shadow price is within the concealment region, \( x \in (\tilde{x}, \hat{x}) \), the asset’s valuation will remain stale at \( \bar{P} \) and book value will remain constant. In figure 1, stale values occur for shadow price paths in between the solid and dashed horizontal lines. Those parts of the shadow price path that remain unreported on the balance sheet have been shaded.

For \( x \in (\tilde{x}, \hat{x}) \), if \( x \) reverts back to \( \hat{x} \) before hitting the point of reckoning \( \hat{x} \), the institution will restart trading without any penalty. Reported valuations will start moving smoothly again in line with the shadow price path. If \( x \) however hits the point of reckoning \( \hat{x} \) before reverting, the institution will voluntarily liquidate its position and exit the market. The reported valuation path will exhibit a jump of size \( \Delta P = \bar{P} - \hat{P} < 0 \). This situation is depicted close to year 8 in figure 1. Once the shadow price path hits the dashed horizontal line, the institution liquidates, leading to a large jump in reported valuation.

We let the bank optimize over \( (\tilde{x}, \hat{x}) \). Let a superscript \( s \), i.e. \( y^s \), denote parameters referring to the case of the constrained institution using possibly stale prices for its accounting valuation. Additionally, let \( \gamma \), i.e. \( \gamma \), denote functions and parameters on the no-trading region.

**Proposition 3** The value function for the constrained problem with the option to mark-to-model is

\[
\tilde{g}(x) = \begin{cases} 
Ax^2 + Bx + C + c^M_M(x) + c_U(x) & x \in [\max(\tilde{x}, \bar{x}), 0] \\
\hat{B}x + \hat{C} + \tilde{c}_M^M(x) + \tilde{c}_U^M(x) & x \in [\min(\hat{x}, \tilde{x}), \max(\hat{x}, \bar{x})] \\
\tilde{g}_p(x) + \tilde{c}_M^M(x) + \tilde{c}_U^M(x) & x \in [\tilde{x}, \hat{x}] 
\end{cases}
\]
with the coefficients $A, B, C, \bar{A}, \bar{B}, c_U$ as described in Propositions 1 and 2. The values for $c^*_M, \bar{c}^*_M, \tilde{c}^*_M, \bar{c}^*_U$ as well as the point of reckoning $\hat{x}$ and the concealment point $\tilde{x}$ solve a system of non-linear equation presented in the appendix. The analytical form of $\tilde{g}_p(x)$ can also be found in the appendix.

The value function is now made up of up to 3 parts, one each for the unconstrained trading region, the possibly constrained trading region, and the no-trading region. The value function derivation strongly relies on the myopic trading of the institution as induced by logarithmic utility, Lemma 1, the fact that the institution’s action choice is only continuous on $[\hat{x}, 0]$ and the assumption of complete liquidation as penalty for violating $\phi \in [0, \bar{\phi}]$. As dynamic programming yields dynamic consistency, maximizing $\tilde{V}$ w.r.t. $\tilde{x}$ and $\hat{x}$ can be reduced to maximizing $c^*_M$, the only parameter on $[\max(\hat{x}, \bar{x}), 0]$ that is affected by the institution’s strategy $(\hat{x}, \tilde{x})$.

**Analysis of the value function.** As the value function is in analytic form, we are now in a position to evaluate it numerically. A priori, we note that most of the action will come from $\hat{x}$ as opposed to $\tilde{x}$ – this is because $\hat{x}$ lies in probabilistically much more likely areas than $\tilde{x}$, and thus the value function will be much more sensitive to small changes to $\hat{x}$ than to $\tilde{x}$. We will use this insight for faster numerics when computing the value function around $x = 0$.

Figure 2 shows an optimal solution in our benchmark case for $\bar{\phi} = 1.6$, with $\tilde{g}$ depicted in the top panel and the corresponding optimal actual leverage $\tilde{\phi}^*$ in the bottom panel. As indicated, the two vertical lines give the optimal no-trading boundaries $(\tilde{x}, \hat{x})$. The value function with marking-to-model $\tilde{g}$ lies unambiguously above $\bar{g}$ on the continuous trading part $x \in [\hat{x}, 0]$ and for large parts of the concealment region $[\tilde{x}, \tilde{x}]$, but of course below the unconstrained value function $g$. The stark decline in the value function $\tilde{g}$ occurs as the institution’s exposure $\tilde{\phi}$ starts increasing rapidly near the pole $\bar{x}$, as can be seen in the bottom panel. Our earlier intuition is visible: the institution trades off better approximating the optimal exposure $\phi^*$ via $\tilde{\phi}$ against a shorter expected horizon $\mathbb{E} [\tau] = \mathbb{E} [\min \{\tau_\rho, \tau_\tilde{x}\}] < \mathbb{E} [\tau_\rho]$ induced by a possible liquidation at $\tilde{x}$, where $\tau_\tilde{x} = \inf \{t : x_t = \tilde{x}\}$.

Consider the almost triangular shaped area in the top panel formed by $\tilde{x}$ to the left, $\bar{g}$ to the top and $\tilde{g}$ to the right, i.e. the region in which $\tilde{g}$ lies below $\bar{g}$. In this region, the institution regrets its past action of having stopped trading at $\hat{x}$. This regret region arises from the assumption that the institution is forced out of the market in case it reveals itself to be in violation of the leverage constraint. The institution optimally decided to stop trading at $\hat{x}$ because the mean reverting property of $x$ resulted in little probability mass on paths hitting $\tilde{x}$ before reverting back to $\hat{x}$ – the possible event of voluntarily liquidation was remote. However, as a ‘bad’ path is realized and

\[12\] The time 0 value to be gained from manipulation is bounded by the difference between $g$ and $\tilde{g}$. 20
Figure 3: Required proportional wealth compensation $\beta$ for losing option of stale valuations in case of benchmark parameters $\kappa = .8, \sigma = .4$ and $\rho = 1/8$.

$x$ moves closer to $\tilde{x}$, the probability of hitting $\tilde{x}$ and therefore of liquidation rises. By continuity, the value function thus starts declining for $x$ close to $\tilde{x}$ as $\bar{g}$ has to converge towards the 'outside option' of 0.

A graphical analysis of the no-trading region can be found in the appendix. The size of the no-trading interval $|\tilde{x} - \hat{x}|$ is decreasing in $\bar{\phi}$. There is almost no effect of small perturbations of $\rho$ on the no-trading boundaries. Only large shifts in $\rho$ affect the no-trading boundaries: as $\rho \to 0$, clearly it is never optimal to liquidate voluntarily. Consider a previously optimal no trading boundary $\hat{x}$. This boundary was optimal for a specific $\rho$, and the agent traded off the possible mean-reversion time of the process against being caught (i.e. the Poisson process hits) with too large leverage. As $\rho$ increases, the probability of being caught with too large leverage increases, whereas the other decision parameters based on $x$ remain unchanged.

Additional intuition can be gained from considering the horizon distributions of $\log W$ under different regimes. The value functions are simply the means of these distributions. Any distribution that allows higher leverage (i.e. the unconstrained or Level 3 via marking-to-model cases) than in the constrained case will result in a higher mean of horizon $\log W$ as the mean-reversion property can be better exploited. But higher leverage also results in thicker left tails – if the horizon (i.e. $\tau_\rho$ or $\tau_{\tilde{x}}$) realizes before prices rebound, the agent’s higher leverage will have resulted in a greater loss compared to the constrained case. In other words, the constraint acts as an 'insurance' against price paths that drop significantly up $\tau_\rho, \tau_{\tilde{x}}$. However, this 'insurance' is not desirable for the agent in that he is willing to take on more leverage.

Value of marking-to-model. Let us now ask how much more wealth at $t = 0$ would an institution require to give up the Level 3 option of marking-to-model? The functional form of
the value function \( \tilde{g} \) has not changed compared to \( g \) on the trading region \([\hat{x}, 0]\). Therefore, we simply have to compare \( c^x_M M(x) \) to \( c^c_M M(x) \). Consider raising the institution’s wealth by a factor \( \beta \). By simple arithmetic, the \( \beta \) that makes the institution indifferent between having Level 3 discretionary accounting and having continuously updated valuations satisfies

\[
\tilde{V}(\beta W_0, x_0) = \tilde{V}(W_0, x_0) \iff \beta = \exp(c^x_M - c^c_M) > 1
\]

Figure 3 presents the numerical solutions for \( \beta \) for our benchmark parameters \( \kappa = 0.8, \sigma = 0.4 \) and \( \rho = 1/8 \). As the hump shape of the graph indicates, there are 2 counteracting forces at work.

As \( \tilde{\phi} \) increases, the probability of the constraint binding becomes more remote. The constrained value function \( \tilde{g} \) will asymptote towards the unconstrained value function \( g \) as \( \tilde{\phi} \to \infty \). Therefore, the additional value to be gained from discretionary accounting has to vanish as well, as \( \tilde{g} \) lies between \( g \) and \( \bar{g} \) on the continuous trading region \([\hat{x}, 0]\).

On the other hand, for \( \tilde{\phi} \) close to 1, there is a large difference between \( g \) and \( \tilde{g} \) even for \( x \) close to 0. However, at \( \tilde{\phi} = 1 \), the discretionary accounting option does not enlarge the strategy space: if the institution stops trading at \( \tilde{\phi} = 1 \), its actual and reported leverage will always be constant at 1, as debt is zero. By continuity, the possible value gain is small for \( \tilde{\phi} \) close to 1: for a given \( \hat{x} \), \( \tilde{\phi} \) remains flat for a large range of \( x \), resulting only in marginal improvements in approximating \( \phi^* \). As \( \tilde{\phi} \) increases, \( \tilde{\phi} \) becomes steeper, allowing the institution to closer approximate \( \phi^* \) and leading to more value to be gained from stale prices.

The accounting option is worth the most for intermediate values of \( \tilde{\phi} \) around 1.5. The institution will require up to 25% more wealth to give up the discretionary accounting option afforded by Level 3 assets in our benchmark case.

4 Extension: Multiple bank interaction

The previous case described the incentives and optimal strategies of a single bank. Consider instead the case in which there are \( n \) banks that each have to acknowledge each other’s transaction prices and have possibly different leverage constraints \( \tilde{\phi} \). If there is continuous price updating, there is no strategic interaction between the banks, as we take the price process as exogenous. With price impact and continuous updating, we would be in the predatory trading case treated by Brunnermeier and Pedersen (2005). In their model, the driving mechanism for predatory trading is the round-trip benefit in which a predatory seller sells for more on the way down and then buys back for less. This mechanism is absent here by design as the underlying price process is exogenous – there is no round-trip benefit. The only impact of a trade is the endogenous
realization of the underlying price. Strategic interaction arises in markets for level 3 assets, as prices are not continuously updated and a bank’s trade imposes transaction prices on competitors.

Without loss of generality, consider the two bank case with banks A and B, as only the least constrained institution and most constrained institution are pivotal. I will assume for tractability that the horizon times of the banks are the same, i.e. $\tau_A = \tau_B = \tau$ and are driven by an intensity $\rho$. However, the banks are facing different leverage constraints and bank A is less constrained than bank B, i.e. $\bar{\phi}_A > \bar{\phi}_B$. What then are possible outcomes of the trading game of the agents?

First, there is an always-trading equilibrium in which all banks trade continuously. The optimal response to an always-trading player is to always trade as well – stopping trading at the constraint will simply lead to an immediate violation of the constraint that leads to liquidation, forfeiting the option payoff of staying in the market.

Second, there can be an equilibrium in which trading stops. Consider the situation in which bank A, the least constraint bank, decides to follow its single player optimal strategy and stops trading on $(\tilde{x}_A, \hat{x}_A)$. Assume further that we are on the monotonically downward sloping part of the curve for $\hat{x}(\bar{\phi})$ (see figure 5 in the appendix) – i.e. a more lax leverage constraint (higher $\bar{\phi}$) leads to an individually optimal lower point of concealment (lower $\hat{x}$).

Bank B now has two options: it can either (1) always-trade and get value $\tilde{g}$ or it can (2) stop trading simultaneously with bank A and get some value $\tilde{g}$ (that is not equal to $\tilde{g}$ as the no-trading region $(\tilde{x}_A, \hat{x}_A)$ is not individually optimal for bank B). If bank B is much more constrained than bank A, it will have a significantly lower leverage at the point when A stops trading. B will then have to weigh if it is worth to stop trading, given that it will be forced to liquidate at $\tilde{x}_A$ – it would be willing to stay in the market longer at $\tilde{x}_A$ (as its implicit leverage is $\bar{\phi}_B < \bar{\phi}_A$ from equation (14)), but A’s liquidation reveals the violation of the leverage constraint on B’s books.

To summarize, bank B will find it optimal to stop trading if its exposure at $\tilde{x}_A$ is close to bank A’s. However, when bank B’s exposure is significantly lower than bank A’s at $\tilde{x}_A$, it will find it optimal to simply keep trading. Looking at the mirror situation, incentives are never aligned when bank B decides to stop trading first. As bank A would like to trade for a bit longer before stopping trading, it can force bank B to keep trading.

Alternatively put, banks’ positions define their incentives to stop trading. When positions lie closely together, incentives to stop trading are aligned and there exists an equilibrium that yields no-trading. The no-trading interval is determined by the least constrained bank, but the most constrained bank will determine if this equilibrium actually exists. When the no-trading equilibrium exists, it is the Pareto optimal one from the view of the firms.

**Proposition 4** Let there be $n$ banks with different leverage constraints $\bar{\phi}$ but with the same horizon time $\tau$. These banks have to acknowledge each other’s transaction prices for accounting purposes. Further, assume that all $\bar{\phi}_i$’s are high enough so that the individually optimal conceal-
ment points of the different banks $\hat{x}_i$ lie on the monotonic downward sloping part of the no-trading interval, i.e. higher $\bar{\phi}_i$'s correspond to lower $\hat{x}_i$'s.

(i) There always exists an always-trading Nash equilibrium in which all banks trade continuously. 
(ii) There can exists a second, no-trading Nash equilibrium when the leverage constraints of the least and most constrained bank lie close enough together. The no-trading interval will be determined by the least constrained bank, and the least constrained bank will still find it optimal to follow. This second equilibrium Pareto dominates the always-trading equilibrium from the banks' perspective.

As most major banks carried sizable positions in related level 3 assets into the current crisis, their incentives to stop trading were aligned. Thus, the model provides some explanation of why banks might refuse to trade. Additionally, the current situation shows that these balance sheet incentives can outweigh possible predatory trading incentives that are abstracted from in this current model.

5 Valuing toxic assets

Takeovers, forced mergers or refinancing at short notice involving companies heavily invested in Level 3 assets pose a formidable challenge to any acquirer or outside investor. Although the target company will open its books, outside investors often lack the necessary expertise to accurately value some of the assets held by the target. As our model showed, it is likely that book valuations based on stale prices are out of line with current shadow prices. As an example, the Korean Development Bank was in such a situation during its talks over a controlling stake at Lehman Brothers shortly before the latter’s demise. Given the rapidly closing window of opportunity and the consequently very tight time frame, the Koreans ultimately passed on investing in Lehman Brothers, partially because of their difficulty appraising some of Lehman Brothers assets.

Our model can give some guidance in how to estimate the market price of such assets to an outsider with only access to the internal transaction prices and book valuations. Recall that we assumed that only institutions active in the market observe the shadow price process. With the derived no-trading boundaries, we are now in a position to examine the expected shadow price of non-traded assets to an outsider.

Our aim is to answer the following question: After prices have become stale, what is the expected shadow price given $t$ units of time have passed since the last trade? We are led to the following corollary that allows us to express the expected price as a conditional expectation.

**Corollary 2** Suppose the asset last traded at time $s$ at a price $P_s$. If the asset has not been
traded on \([s, s + t]\), the expected value of the asset at time \(s + t\) is

\[
E_{s+t}^o [P_{s+t}] = E_s [\exp (x_{s+t}) | x_r \in [\hat{x}, \hat{x}] \text{ for all } r \in [s, s + t]]
\]

where we determine \(\hat{x}\) by \(P_s = \exp (\hat{x})\) and \(E_t^o\) denotes an expectation w.r.t. the outsider information set and accounting information provided by the insider.

The last price at which significant amounts of the asset traded, \(P_s\), is observable from the company’s books, as is the time since last trade \(t\). As there is no known closed form solution to this conditional expectation, we will have to rely on Monte Carlo methods to derive expected price paths. A full-fledged examination of the behavior of \(E_{s+t}^o [P_{s+t}]\) is beyond the scope of this paper.

To recapitulate, our benchmark parameters are \(\kappa = 0.8\), \(\sigma = .4\) and \(\rho = 1/8\), resulting in an average shadow price of 80 cent on the dollar and expected time for this average shadow price to revert back to fundamental value of 5 months.

Figure 4 shows several such expected price paths. The paths are monotonically decreasing in time: the longer an asset has been non-traded, the lower on average its price will be. Although seemingly intuitive, this property is a surprising result in the light of the fact that the underlying shadow price process is mean-reverting. It is the property of the conditioning set \([\hat{x}, \hat{x}]\) that leads to strong enough 'bad news' conditioning to reverse the drift of the price process. Intuitively, 'good' paths that revert back to \(\hat{x}\) before time \(s + t\) will not be included in the conditioning
set, as they result in the institution trading on \((s, s + t)\) – we are dropping the strongest ‘mean-reverting’ paths. The expected price path is further convex in \(t\), i.e. it decreases very fast at the beginning, but the rate of the decrease diminishes. Essentially, the information content of the asset staying toxic for another \(\Delta t\) periods vanishes for assets that have not been traded for some large \(t\).

It is important to note that the model is driven by the static balance sheet constraints that relies on self-reported transaction prices – the bank games the accounting rules to optimally breach this constraint. If the constraint were flexible, in the sense that it would take the above derived decreasing price path when no trade occurs into account, then the no-trade outcome unravels.

The absolute discount from the last trading price, i.e. \(P_s - \mathbb{E}_{s,t}^\sigma [P_{s+t}]\), is relatively constant across different values of \(\kappa\) – the different price paths are approximately constant distance apart. The institution optimally changes its no-trading interval \([\tilde{x}, \hat{x}]\) for different values of \(\kappa\) in such a way that holds the expected discount constant. This is an important result, as it allows outsiders to approximately predict the discount on the reported trading price without detailed knowledge of \(\kappa\).

6 Conclusion

This paper examined the dynamic effects of fair value accounting on the trading behavior of financial institutions and its implications for the valuation of non-traded assets on the institution’s books. Certain OTC markets are so opaque that an institution under marking-to-market accounting only has to acknowledge self-generated transaction prices. Consequently, there can be incentives for balance sheet constrained institutions active in such markets to obstruct price discovery by keeping assets off the market.

We assumed a mean-reverting shadow price as reflecting temporary price depressions due to funding constraints of other market participants. We then solved a combined optimal asset allocation and optimal stopping problem under a leverage constraint. The technical difficulty stemmed from the fact that we were solving for two optimal stopping points simultaneously, the concealment point and the point of reckoning. The resulting analytic form of the value function was then examined numerically.

This generated the following results: (i) The institution trades continuously up to an optimal stopping point where prices have dropped low enough. At this concealment point, it will stop trading to conceal any further losses to its balance sheet. The intuition for the optimality of the concealment point derived from the value of locally relaxing the leverage constraint. With trading suspended, however, the institution’s risk exposure can potentially become excessive,
and the institution optimally liquidates at a point of reckoning. The intuition for the optimal point of reckoning derived mainly from the risk-aversion of the institution in avoiding states of low wealth. (ii) The institution’s book value is updated continuously during the trading region, but will reflect stale values once trading stops. This results in a decoupling of the reported value from the underlying shadow prices, leading to an inflated balance sheet and a relaxed leverage constraint. (iii) This no-trading region can exist even in the case when multiple banks have to acknowledge each other prices. This will be the case when the leverage constraints of the banks lie close to each other. (iv) An outsider evaluating a company’s books cannot observe the current shadow price in the market. With knowledge of the optimal no-trading region and the reported price path, however, he is able to derive an expected shadow price of the non-traded assets. Surprisingly, the expected liquidity discount is increasing and convex in time since last trade and relatively robust to misspecification of the mean-reversion parameter of the shadow price process.

References


Appendices

A Properties of the Kummer Functions $M$ and $U$

This section will summarize important properties of the Kummer Functions $M$ and $U$ as found in Chapter 13 of Abramowitz and Stegun (1972).

**Definition 1** Consider the following ordinary differential equation

$$zy''(z) + (b - z)y'(z) - ay(z) = 0$$

This is Kummer’s equation. Two linearly independent solutions to this ODE are the series solution

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

where $(a)_n = a(a+1)...(a+n-1)$, often called $\text{$_1F_1$}$, and

$$U(a, b, z) = \pi \sin(\pi b) \left[ \frac{M(a, b, z)}{\Gamma(1 + a - b) \Gamma(b)} - z^{-1} \Gamma(1 + a - b, 2 - b, z) \right]$$

**Lemma 2** $M(a, b, z) > 0$ and $U(a, b, z) > 0$ for $z \geq 0, a > 0, b > 0$.

**Proof.** By series definition of $M$, the first part of the lemma follows by inspection. Note further that an alternative representation of $U$ is

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt$$

As all the elements on the RHS are positive, the second part of the lemma follows. ■

**Result 1** For $z \to 0$, $M(a, b, z)$ and $U(a, b, z)$ behave as

$$M(a, b, z) = 1 + O(z)$$

$$U\left(a, \frac{1}{2}, z\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a + \frac{1}{2}\right)} + O\left(z^{\frac{1}{2}}\right)$$

$$U\left(a, \frac{3}{2}, z\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(a)} z^{-\frac{3}{2}} + O(1)$$

**Result 2** Derivatives of $M$ and $U$

$$\frac{d^n}{dz^n} M(a, b, z) = \frac{(a)_n}{(b)_n} M(a + n, b + n, z)$$

$$\frac{d^n}{dz^n} U(a, b, z) = (-1)^n (a)_n U(a + n, b + n, z)$$

Note that $M(a, b, 0) = \frac{\alpha}{\beta}$ and $M(a, b, z) > 0$ for all $z \in (0, \infty)$.

**Result 3** Recursion properties of $M$ and $U$

$$bM(a, b, z) = (b - z) M(a + 1, b + 1, z) + \frac{a + 1}{b + 1} z M(a + 2, b + 2, z)$$

$$U(a, b, z) = (z - b) U(a + 1, b + 1, z) + (a + 1) z U(a + 2, b + 2, z)$$

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The Wronskian of two functions $f_1$ and $f_2$ is defined as $\text{Wr}(f_1, f_2) = f_1 f_2' - f_1' f_2$. Denote by $\text{Wr} = \text{Wr}\{M, U\}$. Then the following property holds for Kummer functions $M$ and $U$

$$\text{Wr}(a, b, z) = -\frac{\Gamma(b)}{\Gamma(a)} z^{-b} e^z$$  \hspace{1cm} \text{(A.4)}

The Gamma function $\Gamma(z)$ is positive for $z > 0$, has a minimum at $z_{\text{min}} = 1.46163...$, and is monotone to both sides of $z_{\text{min}}$. Furthermore, it has the following recursion property:

$$\Gamma(z + 1) = z\Gamma(z)$$

B Proofs

B.1 Collection of special properties of $M$, $U$ when $b = \frac{1}{2}$ and $z = \frac{\kappa}{\sigma^2} x^2$

We will use the following notation: $M(x) \equiv M\left(\frac{\rho^2}{\kappa^2}, \frac{1}{2}, \frac{a}{2}, \kappa x^2, \sigma^2\right)$ and $U(x) \equiv U\left(\frac{\rho^2}{\kappa^2}, \frac{1}{2}, \frac{a}{2}, \kappa x^2, \sigma^2\right)$. We derive the following Lemmata:

Lemma 3 The Wronskian of $M(x)$ and $U(x)$ is given by

$$\text{Wr}(x) = -\text{Sign}[x] \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(a)} 2\sigma \sqrt{\kappa} \exp\left(\frac{\kappa x^2}{\sigma^2}\right)$$

We note that $1/\text{Wr}(x)$ resembles a Normal density, as

$$\frac{1}{\text{Wr}(x)} = -\text{Sign}[x] \frac{\Gamma(a)}{\Gamma\left(\frac{1}{2}\right)} 2\sigma \sqrt{\kappa} \exp\left(-\frac{\kappa x^2}{\sigma^2}\right)$$

Finally, note that we have the following relationship for $1/\text{Wr}(x)$

$$\left(\frac{1}{\text{Wr}(x)}\right)' = -\frac{\partial_z}{\partial x} \frac{1}{\text{Wr}(x)}$$

Proof. Plugging in $z = \frac{\kappa x^2}{\sigma^2}$ and $b = \frac{1}{2}$, taking derivatives and applying the chain rule we get

$$\text{Wr}(x) = M(x) U'(x) - M'(x) U(x)$$

$$= \left[M\left(a, \frac{1}{2}, \frac{a}{2}, \kappa x^2, \sigma^2\right) U\left(a, \frac{1}{2}, \kappa x^2, \sigma^2\right) - M\left(a, \frac{1}{2}, \kappa x^2, \sigma^2\right) U\left(a, \frac{1}{2}, \kappa x^2, \sigma^2\right) \right] \frac{\partial_z}{\partial x}$$

$$= -\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(a)} \frac{\kappa}{\sigma^2} x^2 \exp\left(\frac{\kappa x^2}{\sigma^2}\right) 2\sigma \sqrt{\kappa} \exp\left(\frac{\kappa x^2}{\sigma^2}\right)$$

where we used the chain rule from line 1 to 2, equation (A.4) from line 2 to line 3.

Lemma 4 For $x \leq 0$

$$\lim_{x \to 0} U'(x) = \frac{2\sqrt{\kappa}}{\sigma} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$\lim_{x \to 0} M'(x) = 0$$

i.e. $M'$ vanishes at 0, whereas $U'$ does not.
Proof. Writing out $U'(x)$ and applying results (A.1), we get
\[
\lim_{x \to 0} U'(x) = \lim_{x \to 0} -\frac{\rho}{2\kappa} U \left( \frac{\rho}{2\kappa} + 1, \frac{3}{2}, \frac{\kappa x^2}{\sigma^2} \right) \frac{2\kappa x}{\sigma^2}
\]
\[
= \lim_{x \to 0} -\frac{\rho}{\sigma^2} \left[ \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{\rho}{2\kappa} + 1 \right)} \left( \frac{\kappa x^2}{\sigma^2} \right)^{-\frac{1}{2}} + O(1) \right]
\]
\[
= -\frac{\rho}{\sigma^2} \text{Sign}[x] \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{\rho}{2\kappa} + 1 \right)} \left( \frac{\kappa}{\sigma^2} \right)^{-\frac{1}{2}}
\]
\[
= \frac{\rho}{\sqrt{\kappa \sigma^2}} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{\rho}{2\kappa} + 1 \right)} = \frac{2}{\sqrt{\kappa}} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{\rho}{2\kappa} \right)}
\]
where $\text{Sign}[x] = -1$ for $x \leq 0$ and we used the recursion property of the Gamma function as presented in Result 5. Similarly, for $M'(x)$ we get
\[
\lim_{x \to 0} M'(x) = \lim_{x \to 0} \frac{\rho}{\kappa} M \left( \frac{\rho}{2\kappa} + 1, \frac{1}{2}, \frac{\kappa x^2}{\sigma^2} \right) \frac{2\kappa x}{\sigma^2}
\]
\[
= \lim_{x \to 0} \frac{\rho}{\kappa} \left[ 1 + O(z) \right] \frac{2\kappa x}{\sigma^2} = 0
\]

Lemma 5
\[
\frac{\partial z}{\partial x} U'(x) - U''(x) = -\frac{2\rho}{\sigma^2} U(x)
\]  \hspace{1cm} (A.6)

Proof. Writing out $\frac{\partial z}{\partial x} U'(x) - U''(x)$, noting that $a = \frac{\rho}{2\kappa}$ and $b = \frac{1}{2}$ we see that
\[
\frac{\partial z}{\partial x} U'(x) - U''(x) = -a U(a + 1, b + 1, z) \left( \frac{2\kappa x}{\sigma^2} \right)^2 + a \frac{2\kappa x}{\sigma^2} U(a + 1, b + 1, z)
\]
\[
= -a(a + 1) \left( \frac{2\kappa x}{\sigma^2} \right)^2 U(a + 2, b + 2, z)
\]
\[
= -\frac{2\rho}{\sigma^2} U(x)
\]
where we used the recursion property of $U$ as presented in Result 3. "

Lemma 6
\[
\frac{\partial z}{\partial x} M'(x) - M''(x) = -\frac{2\rho}{\sigma^2} M(x)
\]  \hspace{1cm} (A.7)

Proof. The proof follows the Result 3 the same way as the previous proof. "

B.2 Solutions to the ODE

Proposition 5 Consider the following ODE on a closed convex interval $x \in K \subseteq \mathbb{R}$.
\[
\frac{1}{2} a^2 g''(x) - \kappa x g'(x) - \rho g(x) + q(x, y) = 0
\]
where $q(x, y)$ denotes the particular part for some arbitrary constant $y$. It has the following solution:
(i) The homogeneous solution denoted by the subscript $h$ has the form
\[
g_h(x) = c_M M \left( \frac{\rho}{2\kappa}, \frac{1}{2}, \frac{\kappa x^2}{\sigma^2} \right) + c_U U \left( \frac{\rho}{2\kappa}, \frac{1}{2}, \frac{\kappa x^2}{\sigma^2} \right)
\]
for arbitrary constants $c_M,c_U$. The Kummer parameters are thus $a = \frac{x}{x+y}, b = \frac{1}{2}, z = \frac{a^2}{x+y}$.

(ii) The particular solution denoted by the subscript $p$ has two possible forms:

(ii.a) For $q(x,y) = ax^2 + bx + c$, we have

$$g_p(x) = Aax^2 + BBx + CC$$

where $AA = \frac{a}{2^{a-1}}, BB = \frac{b}{x+y}$ and $CC = \frac{a^2 + c}{x+y}$.

(ii.b) For $q(x,y) \neq ax^2 + bx + c$, we have for some arbitrary limit of integration $l \in K$

$$g_p(x,y) = -\frac{2}{\sigma^2} \left[ \int_x^l q(s,y) \frac{M(x)U(s) - U(x)M(s)}{W_r(s)} \, ds \right]$$

**Proof.** (i) Consider the change of variable $z = \frac{ax^2}{\sigma^2}$ such that $g(x) = v(z)$. Plugging in $v(z)$, we get

$$zy''(z) + \left( \frac{1}{2} - z \right)y'(z) - \frac{\rho}{2\kappa}y(z) = 0$$

for $z \in (0,\infty)$. From the discussion of Kummer functions in the previous part of the appendix, we know two linearly independent solutions of this ODE are $M(x) = M\left(\frac{x}{2\kappa}, \frac{1}{2}, \frac{a^2}{x+y}\right)$ and $U(x) = U\left(\frac{x}{2\kappa}, \frac{1}{2}, \frac{a^2}{x+y}\right)$, as $Wr(x) = -\text{Sign}[x] \frac{1}{\Gamma(1/2) \frac{a^2}{x+y}} \exp\left(\frac{a^2}{x+y}\right) \neq 0$ for all $x \neq 0$.

(ii.a) Simply plugging in $q(x)$ into the ODE and matching coefficients, we get the result.

(ii.b) By the method of variation of coefficients, we can write the particular solution of an ODE in integral form involving its linearly independent solutions (see for example Coddington (1961), Ch.3, Sec. 10, Thm. 11). In our case, the second order ODE has linearly independent solutions $M(x)$ and $U(x)$, and the solution will be of the stated form. Note that $l \in K$ is completely free, and does not influence the solution. ■

**Lemma 7** For the variation of coefficients solution

$$g_p(x,y) = -\frac{2}{\sigma^2} \left[ \int_x^l q(s,y) \frac{M(x)U(s) - U(x)M(s)}{W_r(s)} \, ds \right]$$

the function and its first derivative w.r.t. $x$ vanish at the limit of integration $l$

$$g_p(l) = \frac{\partial g_p(l,y)}{\partial x} = 0$$

but the second derivative w.r.t. $x$ does not

$$\frac{\partial^2 g_p(l,y)}{\partial x^2} = -\frac{2}{\sigma^2}q(l,y)$$

**Proof.** Taking derivatives, we see that

$$\frac{\partial g_p(l,y)}{\partial x} = -\frac{2}{\sigma^2} \left[ \int_x^l q(s,y) \frac{M'(x)U(s) - U'(x)M(s)}{W_r(s)} \, ds \right]$$

$$\frac{\partial^2 g_p(l,y)}{\partial x^2} = -\frac{2}{\sigma^2} \left[ \int_x^l q(s,y) \frac{M''(x)U(s) - U''(x)M(s)}{W_r(s)} \, ds \right]$$

Plugging in $x = l$, the integrals vanish, and we are left with the result. ■
B.3 Optimality proofs

Notation: let $c_M, c_U$ denote the coefficients on $M$ and $U$ in the unconstrained region, $\bar{c}_M, \bar{c}_U$ in the constrained region without hiding, and $\tilde{c}_M, \tilde{c}_U$ in the constrained region with hiding.

**Proof.** [Proof of Proposition 1] Let us recall equation (9):

$$\frac{1}{2} \sigma^2 g''(x) - \kappa x g'(x) - \rho g(x) + \max_\phi \left\{-\kappa x + \frac{1}{2} \sigma^2 \phi^2 \right\} = 0$$

We note that the particular part of the ODE is generated by the drift of the log wealth. The optimal investment policy is easily derived from the FOC on $(-\infty, 0)$:

$$\phi^* = \left\{-\frac{-\kappa x + \frac{1}{2} \sigma^2 \phi^2}{\sigma^2}\right\} 1_{\{x \neq 0\}}$$

Plugging this back in, we see that the ODE becomes

$$\frac{1}{2} \sigma^2 g''(x) - \kappa x g'(x) - \rho g(x) + \left[-\frac{-\kappa x + \frac{1}{2} \sigma^2 \phi^2}{\sigma^2}\right]^2 = 0$$

By Proposition 5 part (i) and (ii.a), we know that this equation is solved by

$$g(x) = c_M M(x) + c_U U(x) + Ax^2 + Bx + C_{g_p(x)}$$

where

$$A = \frac{\kappa^2}{2\sigma^2(2\kappa + \rho)} \quad (A.9)$$

$$B = -\frac{\kappa}{2(\kappa + \rho)}$$

$$C = \frac{\kappa^2}{2(2\kappa + \rho)} + \frac{\sigma^2}{8\rho}$$

As we assume that the process $x$ reflects from below at 0, we have as a boundary condition $g'(0) = -\phi(0)$ from equating the $dK$ terms on the Ito expansion of $V = \log W + g(x)$ (see Dixit (1993) for an intuitive explanation). We defer showing $\phi(0) = \bar{\phi}$ is optimal until the verification. Thus

$$0 = \lim_{x \to 0} \left\{c_M M'(x) + c_U U'(x) + g_p'(x)\right\}$$

$$g_p'(0) = -c_U 2\sqrt{\kappa} \frac{\Gamma\left(\frac{1}{2}\right)}{\sigma} \frac{\Gamma\left(\frac{\rho}{2\kappa}\right)}{\Gamma\left(\frac{\rho}{2}\right)}$$

where we used (A.5) and the recursion property of the Gamma function. Noting that we are approaching 0 from below, $c_U$ is completely determined by the boundary condition:

$$c_U = -\frac{\sqrt{\kappa} \sigma}{\rho} \frac{\Gamma\left(\frac{\rho}{2\kappa} + 1\right)}{\Gamma\left(\frac{1}{2}\right)} B = -\frac{\sqrt{\kappa} \sigma}{2\kappa} \frac{\Gamma\left(\frac{\rho}{2\kappa}\right)}{\Gamma\left(\frac{1}{2}\right)} B$$

$$= \frac{\sigma \sqrt{\kappa}}{4(\kappa + \rho)} \frac{\Gamma\left(\frac{\rho}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

Show that $c_M = 0$:

Let us impose an exogenous condition that the institution has to liquidate when $x$ reaches a certain threshold $x'$. At this threshold, the value function will be equal to $\log W$. As this assumption does not affect the ODE of the value function away from this boundary, we are still left with the same general solution. Solving for $c_M$, we see that $c_M = \frac{(x')^2 + Bx' + C + c_U U(x')}{M(x')}$ which converges to 0 rapidly as $x' \to -\infty$. Note that this $x'$ dependent
stopping time did not affect the optimal investment policy. Similarly, we could have simply imposed the boundary condition \( \lim_{x \to -\infty} M(x)^{-1} g(x) = 0 \).

Show that \( g(x) \) is decreasing in \( x \):

The quadratic part of the equation \( Ax^2 + Bx + C \) has its minimum at \(-\frac{B}{2A} > 0\), so that it is decreasing for all \( x < 0 \). By \( cU > 0 \) and \( U'(x) > 0 \) and \( U''(x) > 0 \) for all \( x \), we see that the value function is decreasing in \( x \): \( 2Ax + B + cuU'(x) \) is zero at \( 0 \). With \( A > 0 \) and \( U''(x) > 0 \), the result follows.

Verification argument:

We will verify the optimal strategy \( \phi^* \) directly. Once the optimality of \( \phi^* \) is established, what remains is essentially computing an expectation for a fixed \( \phi^* \). For the direct derivation, we will utilize the calculus of variations. Write out the value function as an expectation, i.e.

\[
V(W, x) = \max_{\phi} \mathbb{E} [\log W_x] = \max_{\phi} \mathbb{E} \left[ \int_0^\tau (-\kappa x + \frac{1}{2} \sigma^2) \phi - \frac{1}{2} \sigma^2 \phi'^2 ds + \int_0^\tau \sigma \phi dZ - \int_0^\tau \phi(0) dK \right]
\]

As \( K \) is increasing only on \( \{x = 0\} \) and flat otherwise, the agent optimally sets \( \phi(0) = 0 \), so we can drop the integral w.r.t. \( dK \) (bear in mind that we ignore the possible local time loss generated by the trading strategy). Note that \( \tau \) is independent of \( Z \), so that we can make use of conditional expectations. Also note that \( \tau \) is almost surely finite. We can then write out the expectation as

\[
\max_{\phi} \mathbb{E} [\log W_x] = \max_{\phi'} \mathbb{E}' \left[ E^Z \left[ \int_0^\tau (-\kappa x + \frac{1}{2} \sigma^2) \phi - \frac{1}{2} \sigma^2 \phi'^2 ds \right] + \mathbb{E}^Z \left[ \int_0^\tau \sigma \phi dZ \right] \right]
\]

where \( \mathbb{E}' \) and \( \mathbb{E}^Z \) are expectations w.r.t. \( \tau \) and \( Z \) respectively. By our assumption that \( \phi \) is in \( L^2 \), and by \( \tau \) a.s. finite, we know that the stochastic integral \( \int_0^\tau \sigma \phi dZ \) is a martingale and its expectation is therefore zero. Maximizing the expectation thus reduces to maximizing the expectation of the time integral \( \mathbb{E} [\int \ldots dS] \). As \( \tau \) is independent of \( Z \), we can maximize path-by-path. Applying simple calculus of variations will give \( \phi^* \) as listed in the main part of the paper that we derived from dynamic programming. The value function of the main part can thus be interpreted as the solution to the expectation of \( \mathbb{E} [\log W_x] \) with fixed strategy \( \phi^* \). This verifies that our value functions is indeed optimal. \( \blacksquare \)

**Proof.** [Proof of Corollary 1] First note that

\[
g(0) = C + cuU(0) = \frac{\kappa^2}{2(\kappa + \rho)} + \frac{\sigma^2}{\kappa} + \frac{\sigma \sqrt{\kappa}}{4(\kappa + \rho)} \left( \frac{3}{2\kappa} \right) \Gamma \left( \frac{3}{2\kappa} \right)
\]

Taking derivatives, we are faced with the polygamma function \( \psi_0 \) (\( y \)) in the derivative of \( \Gamma \) (\( y \)): \( \Gamma'(y) = \Gamma(y) \psi_0(y) \).

Note that \( \psi_0(y) - \psi_0(y + \frac{1}{2}) < 0 \) for all \( y > 0 \). Taking derivatives, we have the following:

(i)

\[
\frac{\partial g(0)}{\partial \kappa} = \frac{8\kappa^{3/2} (\kappa + \rho)^3 \Gamma \left( \frac{3}{2\kappa} \right) - \rho (2\kappa + \rho)^2 \sigma \Gamma \left( \frac{3}{2\kappa} \right) [\kappa (\kappa - \rho) + \rho (\kappa + \rho)] \psi_0 \left( \frac{3}{2\kappa} \right) - \psi_0 \left( \frac{3}{2\kappa} + \frac{1}{2} \right) \}}{8\kappa^{3/2} \rho (\kappa + \rho)^2 (2\kappa + \rho)^2 \Gamma \left( \frac{3}{2\kappa} \right) \left( \frac{3}{2\kappa} + \frac{1}{2} \right) \}
\]

The first term in the numerator is clearly positive. Let us concentrate on the second term and rewrite it in terms of \( y = \frac{3}{2\kappa} \)

\[
-\frac{\rho (2\kappa + \rho)^2 \sigma \Gamma \left( \frac{3}{2\kappa} \right) [\kappa (\kappa - \rho) + \rho (\kappa + \rho)] \psi_0 \left( \frac{3}{2\kappa} \right) - \psi_0 \left( \frac{3}{2\kappa} + \frac{1}{2} \right) \}}{8\kappa^{3/2} \rho (\kappa + \rho)^2 (2\kappa + \rho)^2 \Gamma \left( \frac{3}{2\kappa} \right) \left( \frac{3}{2\kappa} + \frac{1}{2} \right) \}
\]

The term in curly brackets is only a function of \( y \) and negative for all \( y > 0 \). As it is multiplied by a negative quantity we conclude with the claim.
(ii) By inspection

$$\frac{\partial g(0)}{\partial \sigma} = \frac{\sigma}{4 \rho} + \sqrt{\frac{\kappa}{2}} \left( \frac{\phi}{\sqrt{\kappa + \rho}} \right) \Gamma \left( \frac{\phi}{\sqrt{\kappa + \rho}} \right) > 0$$

(iii) Noting that $\psi_0 \left( \frac{\phi}{\sqrt{\kappa + \rho}} \right) - \psi_0 \left( \frac{\phi}{\sqrt{\kappa + \rho}} + \frac{1}{2} \right) < 0$, we see that

$$\frac{\partial g(0)}{\partial \rho} = 1 \left( -\frac{\kappa \phi^2 (\kappa + \rho)}{\rho^2} + \frac{\sigma^2}{\rho} \left[ -2 \kappa + \phi \left( \psi_0 \left( \frac{\phi}{\sqrt{\kappa + \rho}} \right) - \psi_0 \left( \frac{\phi}{\sqrt{\kappa + \rho}} + \frac{1}{2} \right) \right) \right] \right) < 0$$

\[ \textbf{Proof.} \text{ [Proof of Proposition 2]} \] Using dynamic programming as in the previous proof, and hypothesizing the same additive separable functional form in the two state-variables, we are led once again to

$$\frac{1}{2} \sigma^2 g''(x) - \kappa x g'(x) - \rho g(x) + \max_{\phi \in [\bar{\phi}, \phi]} \left\{ \left[ -\kappa x + \frac{1}{2} \sigma^2 \right] \phi - \frac{1}{2} \sigma^2 \phi^2 \right\} = 0$$

Note that we are maximizing the drift of log wealth. The constrained maximization is simple, as it does not involve the value function $g$: follow the optimal policy until constrained, then simply trade to stay at the boundary. Mathematically, we have

$$\tilde{\phi}^* (x) = \begin{cases} \tilde{\phi} & x < \bar{x} \\ \phi^* (x) & \bar{x} \leq x \leq 0 \end{cases}$$

$$\bar{x} = -\frac{\sigma^2}{\kappa} \left( \tilde{\phi} - \frac{1}{2} \right)$$

Plugging this back in, we are left with an ODE on 2 regions, one in which $\tilde{\phi}^* (x) = \phi^* (x)$, and one in which $\tilde{\phi}^* (x) = \tilde{\phi}$. To link these 2 regions, we need the homogeneous solutions to the ODE. By the property of the value function being an expectation (an integral over states w.r.t. the probability measure), and the fact that the process can recross $\bar{x}$, we need to match the two solutions’ value and first derivative at $\bar{x}$. Again, for details see Dixit (1993). On the constrained region, the value function follows

$$\frac{1}{2} \sigma^2 \bar{g}''(x) - \kappa x \bar{g}'(x) - \rho \bar{g}(x) + \left( -\kappa x + \frac{1}{2} \sigma^2 \right) \tilde{\phi} - \frac{1}{2} \sigma^2 \tilde{\phi}^2 = 0$$

By Proposition 5 part (i) and (ii.a), we know that this equation is solved by

$$\bar{g}(x) = \bar{c}_M M(x) + \bar{c}_U U(x) + \bar{B} x + \bar{C}$$

for some parameters $\bar{c}_M, \bar{c}_U$ and

$$\bar{B} = -\frac{\kappa \tilde{\phi}}{\kappa + \rho} \tag{A.12}$$

$$\bar{C} = -\frac{\sigma^2 \tilde{\phi} \left( \tilde{\phi} - 1 \right)}{2 \rho}$$

As $\lim_{x \to -\infty} \bar{g}(x) / g(x) < \infty$ and $g(x) \geq 0$, we must have $\bar{c}_M = 0$. $c_u$ (the coefficient on $U$ in the unconstrained region) still has the same value as it is pinned down by an unchanged boundary condition at $x = 0$. Applying value matching and smooth pasting, we are left with the following system of equations

$$\begin{bmatrix} M'(\bar{x}) & -U'(\bar{x}) \\ M''(\bar{x}) & -U''(\bar{x}) \end{bmatrix} \begin{bmatrix} \bar{c}_M \\ \bar{c}_U \end{bmatrix} = \begin{bmatrix} \bar{g}_x(\bar{x}) - g_x(\bar{x}) - c_u U(\bar{x}) \\ g''(\bar{x}) - \bar{g}_x(\bar{x}) - g''(\bar{x}) - c_u U''(\bar{x}) \end{bmatrix}$$
Noting that the determinant is equal to \(-W_\tau(\hat{x})\), we can write the solution as

\[
\begin{bmatrix}
\hat{c}_M \\
\hat{c}_L \\
\end{bmatrix} = \frac{1}{-W_\tau(\hat{x})} \begin{bmatrix}
U'(\hat{x}) & -U(\hat{x}) \\
M'(\hat{x}) & -M(\hat{x}) \\
\end{bmatrix} \begin{bmatrix}
\hat{g}_p(\hat{x}) - g_p(\hat{x}) - c_UU(\hat{x}) \\
\hat{g}_p(\hat{x}) - g_p(\hat{x}) - c_UU'(\hat{x}) \\
\end{bmatrix}
\]

The verification argument is exactly the same as in the above case, as we note that \(\phi\) bounded immediately implies \(\phi\) is \(L^2\) integrable. All other conditions remain the same. We note that by continuity, \(\hat{\phi} \to \infty\) implies \(\hat{g} \to g\).

**Proof.** [Proof of Proposition 3] As in the previous proofs, we conjecture a value function that is of additive separable form, \(\log W + \hat{g}(x)\). Denote by \(\hat{x}\) the point at which the institution stops trading.

Note first that by the ODE structure, optimal trading is the same as in the previous proposition on \([\hat{x}, 0]\) irrespective of the functional form of \(g\).

On the no-trading region we have exposure following \(\hat{\phi}(x, \hat{x})\) as show in Lemma 1. The log wealth dynamics are

\[
d\log W = \left[\hat{\phi}(x, \hat{x}) \left(-\kappa x + \frac{1}{2}\sigma^2\right) - \frac{1}{2} \hat{\phi}(x, \hat{x})^2 \sigma^2\right] dt + \hat{\phi}(x, \hat{x}) \sigma dZ
\]

We thus have highly non-linear wealth dynamics that give rise to a particular part \(\bar{\mu}_{\log W}(s, \hat{x})\) in the underlying ODE on \(\hat{x} \leq x \leq \bar{x}:

\[
\frac{1}{2}\sigma^2 \hat{g}''(x) - \kappa x \hat{g}'(x) - \rho \hat{g}(x) + \bar{\mu}_{\log W}(x, \hat{x}) = 0
\]

with the particular part being, by Lemma 1

\[
\bar{\mu}_{\log W}(x, \hat{x}) = \left(-\kappa x + \frac{1}{2}\sigma^2\right) \frac{\hat{\phi}(x, \hat{x}) \exp[x - \hat{x}]}{1 + \hat{\phi}(x, \hat{x}) \exp[x - \hat{x}] - 1} - \frac{1}{2} \sigma^2 \left(\frac{\hat{\phi}(x, \hat{x}) \exp[x - \hat{x}]}{1 + \hat{\phi}(x, \hat{x}) \exp[x - \hat{x}] - 1}\right)^2
\]

By proposition 5 part (i) and (ii.b), we know that this equation is solved by

\[
\hat{g}(x) = c_M M(x) + c_U U(x) + \hat{g}_p(x, \hat{x})
\]

\[
\hat{g}_p(x, \hat{x}) = \frac{2}{\sigma^2} \left[\int_\hat{x}^l \bar{\mu}_{\log W}(s, \hat{x}) \frac{U(s) M(s) - M(s) U(s)}{W_\tau(s)} ds\right]
\]

with the arbitrary limit of integration \(l \in [\hat{x}, \bar{x}]\).

Optimality dictates, by Dumas (1991) and Dixit (1993), a smooth pasting condition at \(\bar{x}\). We can also derive this smooth pasting condition from the optimization

\[
\frac{dc_M}{d\hat{x}} = \frac{\partial c_M}{\partial \hat{x}} \frac{d\hat{x}}{\partial \hat{x}} + \frac{\partial c_M}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \hat{x}} = \frac{\partial c_M}{\partial \hat{x}}
\]

where we assumed that \(\hat{x}\) was chosen optimally, so that by the envelope theorem \(\frac{\partial c_M}{\partial \hat{x}} = 0\). The resulting equation takes the form

\[
c_M' M'(\bar{x}) + c_M' U'(\bar{x}) = f(\bar{x}) - \hat{g}(\bar{x}) \tag{A.13}
\]

We need to consider two possible outcomes, (i) the institution stops trading before it reaches \(\bar{x}\), i.e. \(\bar{x} < \hat{x} < 0\) or (ii) the institution stops trading at a point later than \(\bar{x}\), i.e. \(\hat{x} < \bar{x} < 0\). Again, we ignore a possible local time loss due to discontinuities in the trading strategy for tractability.

Consider the resulting value function with optimal liquidation time \(\hat{x}\). On the unconstrained section of the problem, the value function is described by \(Ax^2 + Bx + C + c_M M + c_U U\). On the constrained section on which the agent still continues trading, it is described by \(Bx + C + c_M M + c_U U\). Note that \(c_U\) is uniquely fixed by the reflection point \(x = 0\) for any specification of \(\bar{x}\). Thus, as dynamic programming implies dynamic consistency, maximizing the value function w.r.t. \(\hat{x}\) amounts to maximizing \(c_M\) w.r.t. \(\hat{x}\).

Optimality for \(\hat{x}\), again following Dumas (1991) and Dixit (1993), results in

\[
\frac{dc_M}{d\hat{x}} = \frac{\partial c_M}{\partial \hat{x}} \frac{d\hat{x}}{\partial \hat{x}} + \frac{\partial c_M}{\partial \hat{x}} \frac{d\hat{x}}{d\hat{x}} = \frac{\partial c_M}{\partial \hat{x}}
\]

where we used the envelope theorem \(\frac{\partial c_M}{\partial \bar{x}} = 0\).

In the following, we will be using the following results, where \(g_{\partial t}\) stands for a derivative w.r.t. to the first argument.
etc.

\[
\tilde{g}_p(x, \hat{x}) = \frac{2}{\sigma^2} \left[ \int_x^l \hat{\mu}_{\log W}(s, \hat{x}) U(x) M(s) - M(x) U(s) ds \right] W_r(s)
\]

\[
\tilde{g}_{p2}(x, \hat{x}) = \frac{2}{\sigma^2} \left[ \int_x^l \frac{\hat{\mu}_{\log W}(s, \hat{x})}{\partial \hat{x}} U(x) M(s) - M(x) U(s) ds \right] W_r(s)
\]

\[
\tilde{g}_{p12}(x, \hat{x}) = \frac{2}{\sigma^2} \left[ \int_x^l \frac{\hat{\mu}_{\log W}(s, \hat{x})}{\partial \hat{x}} U'(x) M(s) - M'(x) U(s) ds \right] W_r(s)
\]

\[
\tilde{g}_{p22}(x, \hat{x}) = \frac{2}{\sigma^2} \left[ \int_x^l \frac{\hat{\mu}_{\log W}(s, \hat{x})}{\partial \hat{x}^2} U(x) M(s) - M(x) U(s) ds \right] W_r(s)
\]

and \(\tilde{g}_{p1}(\hat{x}, \hat{x}) = 0\) and \(\tilde{g}_{p11}(\hat{x}, \hat{x}) = -\frac{2}{\sigma^2} \hat{\mu}_{\log W}(\hat{x}, \hat{x})\) as shown in (A.8). Picking \(l = \hat{x}\), we get

\[
\frac{\partial g_p(\hat{x}, \hat{x})}{\partial \hat{x}} = \tilde{g}_{p1}(\hat{x}, \hat{x}) + \tilde{g}_{p2}(\hat{x}, \hat{x}) = 0
\]

\[
\frac{\partial^2 g_p(\hat{x}, \hat{x})}{\partial \hat{x}^2} = \tilde{g}_{p11}(\hat{x}, \hat{x}) + 2\tilde{g}_{p12}(\hat{x}, \hat{x}) + \tilde{g}_{p22}(\hat{x}, \hat{x})
\]

\[
= \tilde{g}_{p11}(\hat{x}, \hat{x}) = -\frac{2}{\sigma^2} \hat{\mu}_{\log W}(\hat{x}, \hat{x})
\]

(i) Consider any path up to \(\hat{x}\). By Lemma 1, the institution will follow the optimal constrained strategy \(\hat{\phi}^*(x)\) up to \(\hat{x}\).

With \(\tilde{x} \leq \hat{x} \leq 0\), the function \(c_M\) is described by smooth pasting at the reflection point \(x = 0\), value matching and smooth pasting at \(\tilde{x}\), and value matching at \(\hat{x}\). The resulting system of equations can be written in matrix form as

\[
\begin{pmatrix}
M(\hat{x}) & -M(\tilde{x}) & -U(\tilde{x}) \\
M'(\hat{x}) & -M'(\tilde{x}) & -U'(\tilde{x}) \\
0 & M(\tilde{x}) & U(\tilde{x})
\end{pmatrix}
\begin{pmatrix}
c_M \\
\tilde{c}_M \\
\tilde{c}_U
\end{pmatrix}
= \begin{pmatrix}
\tilde{g}_p(\hat{x}, \hat{x}) - g_p(\hat{x}) - c_U U(\hat{x}) \\
\tilde{g}_{p1}(\hat{x}, \hat{x}) - g_p'(\hat{x}) - c_U U'(\hat{x}) \\
f(\hat{x}) - \tilde{g}_p(\hat{x}, \hat{x})
\end{pmatrix}
\]

Taking the inverse and substituting in for the Wronskian, we see that \(c_M\) has the form

\[
c_M = \frac{1}{M(\hat{x})} \left[ \frac{U'(x)M(x) - M'(x)U(x)}{W_r(x)} \right] \frac{M(x)U(x) - U(x)M(x)}{W_r(x)} 1 \begin{pmatrix}
\tilde{g}_p(\hat{x}, \hat{x}) - g_p(\hat{x}) - c_U U(\hat{x}) \\
\tilde{g}_{p1}(\hat{x}, \hat{x}) - g_p'(\hat{x}) - c_U U'(\hat{x}) \\
f(\hat{x}) - \tilde{g}_p(\hat{x}, \hat{x})
\end{pmatrix}
\]

Taking derivatives w.r.t. \(\hat{x}\), we get

\[
\frac{\partial c_M}{\partial \hat{x}} = \begin{pmatrix}
\frac{U'(x)M(x) - M'(x)U(x)}{M(x)W_r(x)} \frac{M(x)U(x) - U(x)M(x)}{M(x)W_r(x)} \\
\frac{U'(x)M(x) - M'(x)U(x)}{M(x)W_r(x)} \frac{M(x)U(x) - U(x)M(x)}{M(x)W_r(x)} \\
-\frac{2g_p(\hat{x}) - c_U U(\hat{x})}{M(x)}
\end{pmatrix} \begin{pmatrix}
\tilde{g}_p(\hat{x}, \hat{x}) - g_p(\hat{x}) - c_U U(\hat{x}) \\
\tilde{g}_{p1}(\hat{x}, \hat{x}) - g_p'(\hat{x}) - c_U U'(\hat{x}) \\
f(\hat{x}) - \tilde{g}_p(\hat{x}, \hat{x})
\end{pmatrix}
\]

\[
-\frac{2g_p(\hat{x}) - c_U U(\hat{x})}{M(x)} \begin{pmatrix}
\tilde{g}_p(\hat{x}, \hat{x}) - g_p(\hat{x}) - c_U U(\hat{x}) \\
\tilde{g}_{p1}(\hat{x}, \hat{x}) - g_p'(\hat{x}) - c_U U'(\hat{x}) \\
f(\hat{x}) - \tilde{g}_p(\hat{x}, \hat{x})
\end{pmatrix}
\]

where we used relations (A.6) and (A.7) repeatedly: from line 1 to line 2, and to eliminate \(c_U\) from line 3 to line
4. Recall that we picked the strategy \( \hat{\phi} \) where we used \( \tilde{\phi} \) in matrix form as
\[
\begin{bmatrix}
\tilde{\phi}_{11} (\hat{x}, \hat{x}) - \tilde{\phi}'' (\hat{x}) \\
\frac{-2 \rho}{\sigma^2} (A \hat{x}^2 + B \hat{x} + C) + \frac{2 \kappa}{\sigma^2} \hat{x} (2A \hat{x} + B) - 2A - \frac{2}{\sigma^2} \tilde{\mu} \log W (\hat{x}, \hat{x}) \\
0
\end{bmatrix}
\]
(A.14)
where we used \( A, B, C \) from above. By \( M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) < 0 \) for \( \hat{x} < \hat{x} \) we have the first term negative. We see that at \( \hat{x} = \tilde{x} \) this term becomes zero. Note further that
\[
\tilde{\phi}_{12} (\hat{x}, \tilde{x}) = \frac{2}{\sigma^2} \left[ \int_{\hat{x}}^{\tilde{x}} \frac{1}{\partial x} \tilde{\mu} \log W (s, \hat{x}) U (x) M (s) - M (x) U (s) \right] ds
\]
(A.15)

(ii) Consider any path up to \( \hat{x} \). Once again, by Lemma 1, the institution will follow the optimal constrained strategy \( \hat{\phi}^* (x) \) up to \( \hat{x} \).

With \( \tilde{x} \leq \hat{x} \leq 0 \), the function \( c_M \) is described by smooth pasting at the reflection point \( x = 0 \), value matching and smooth pasting at \( \tilde{x} \) and \( \hat{x} \), and value matching at \( \tilde{x} \). The resulting system of equations can be written in matrix form as
\[
\begin{bmatrix}
M (\hat{x}) & -M (\hat{x}) & -U (\hat{x}) & 0 & 0 \\
M' (\hat{x}) & -M' (\hat{x}) & -U' (\hat{x}) & 0 & 0 \\
0 & M (\hat{x}) & U (\hat{x}) & -M (\hat{x}) & -U (\hat{x}) \\
0 & M' (\hat{x}) & U' (\hat{x}) & -M' (\hat{x}) & -U' (\hat{x}) \\
0 & 0 & 0 & M (\tilde{x}) & U (\tilde{x})
\end{bmatrix}
\begin{bmatrix}
c'_M \\
c''_M \\
c'_U \\
c''_U \\
1
\end{bmatrix} = \begin{bmatrix}
g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) - c_U U (\hat{x}) \\
g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) - c_U U (\hat{x}) \\
g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) \\
g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) - g_{\hat{p}} (\hat{x}) \\
f (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}, \hat{x})
\end{bmatrix}
\]

Taking the inverse and substituting in for the Wronskians, we see that \( c_M \) has the form
\[
c_M = \frac{1}{M (\tilde{x})} \begin{bmatrix}
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M(\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M(\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M(\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
1
\end{bmatrix} \begin{bmatrix}
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x})
\end{bmatrix}
\]

where \( \cdot \) denotes functions unrelated to \( \hat{x} \) that will drop out in the derivative. Taking derivatives, we get almost the same result as in section (i) of the proof
\[
\frac{\partial c_M}{\partial \hat{x}} = \begin{bmatrix}
(M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) M (\hat{x}) - (M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) U (\hat{x}) \\
(M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) M (\hat{x}) - (M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) U (\hat{x}) \\
(M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) M (\hat{x}) - (M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) U (\hat{x}) \\
(M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) M (\hat{x}) - (M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x})) U (\hat{x}) \\
1
\end{bmatrix} \begin{bmatrix}
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x})
\end{bmatrix}
\]

\[
= \begin{bmatrix}
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
1
\end{bmatrix} \begin{bmatrix}
\frac{-2 \rho}{\sigma^2} (A \hat{x}^2 + B \hat{x} + C) + \frac{2 \kappa}{\sigma^2} \hat{x} (2A \hat{x} + B) - 2A - \frac{2}{\sigma^2} \tilde{\mu} \log W (\hat{x}, \hat{x}) \\
\frac{-2 \rho}{\sigma^2} (A \hat{x}^2 + B \hat{x} + C) + \frac{2 \kappa}{\sigma^2} \hat{x} (2A \hat{x} + B) - 2A - \frac{2}{\sigma^2} \tilde{\mu} \log W (\hat{x}, \hat{x}) \\
\frac{-2 \rho}{\sigma^2} (A \hat{x}^2 + B \hat{x} + C) + \frac{2 \kappa}{\sigma^2} \hat{x} (2A \hat{x} + B) - 2A - \frac{2}{\sigma^2} \tilde{\mu} \log W (\hat{x}, \hat{x}) \\
\frac{-2 \rho}{\sigma^2} (A \hat{x}^2 + B \hat{x} + C) + \frac{2 \kappa}{\sigma^2} \hat{x} (2A \hat{x} + B) - 2A - \frac{2}{\sigma^2} \tilde{\mu} \log W (\hat{x}, \hat{x}) \\
1
\end{bmatrix} \begin{bmatrix}
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x}) \\
\tilde{g}_{\hat{p}} (\hat{x}) - \tilde{g}_{\hat{p}} (\hat{x})
\end{bmatrix}
\]

\[
= \begin{bmatrix}
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \\
1
\end{bmatrix} \begin{bmatrix}
\frac{1}{M (\hat{x})} \frac{\partial \tilde{\mu} \log W (\hat{x}, \hat{x})}{\partial \hat{x}}
\end{bmatrix}
\]
Recall that we picked \( l = \hat{x} \) and that \( \hat{g}_p \) is a first order polynomial \( \hat{g}_p (x) = BBx + CC \), we get

\[
\begin{bmatrix}
-\frac{2\kappa}{\sigma^2} \\
-\frac{\sigma}{\sigma^2} \\
1
\end{bmatrix} \begin{bmatrix}
\hat{g}_p (\hat{x}) - \hat{g}_p (\hat{x}) \\
\hat{g}_p' (\hat{x}) - \hat{g}_p' (\hat{x}) \\
\hat{g}_p'' (\hat{x}) - \hat{g}_p'' (\hat{x})
\end{bmatrix}
\]

\[
= \frac{2\rho}{\sigma^2} (\hat{B} \hat{x} + \bar{C}) + \frac{2\kappa}{\sigma^2} \hat{B} \hat{x} - \frac{2}{\sigma^2} \hat{\mu} \log W (\hat{x}, \hat{x})
\]

\[
= 0
\]

where we used \( \bar{B}, \bar{C} \) from above. The sign will therefore be solely determined by

\[
- \frac{1}{M (\hat{x})} \hat{g}_p (x, \hat{x}) = \frac{1}{M (\hat{x})} \frac{2}{\sigma^2} \int \frac{\partial \log W (s, \hat{x})}{\partial \hat{x}} M (x) U (s) - U (x) M (s) ds
\]

Notice that the derivative \( \frac{\partial \hat{M}}{\partial \hat{x}} \) is continuous through \( \hat{x} \).

Combining the results of part (i) and part (ii) of the proof, more specifically equations (A.14), (A.15) and (A.16), we are left with

\[
0 = 1_{(\hat{x} \geq 0)} (M (\hat{x}) U (\hat{x}) - U (\hat{x}) M (\hat{x}) \left[ \frac{2\kappa \hat{x} + \sigma^2 (2\hat{\phi} - 1)}{4\sigma^4} \right]
\]

\[
+ \frac{1}{M (\hat{x})} \frac{2}{\sigma^2} \int \frac{\partial \log W (s, \hat{x})}{\partial \hat{x}} M (x) U (s) - U (x) M (s) ds
\]

Notice that we pick the solution for \( (\hat{x}, \hat{x}) \) that has \( \hat{g} \geq 0 \) everywhere. We know this solution exists from picking \( \hat{x} = \hat{x} \) for arbitrary \( \hat{x} \) results in a positive value function. Shifting out \( \hat{x} \) will clearly result in a positive value function as long as \( \hat{\phi} < \phi^* \). We also know that if \( \hat{x} = \mathbb{F} \), that the value becomes negative infinity.

Verification:

Note that \( \hat{\theta} (x, \hat{x}) \) is potentially unbounded. From Lemma 1, we know that there must exist an \( \varepsilon > 0 \) such that \( \hat{x} > \mathbb{F} + \varepsilon \). We pick an arbitrary but small \( \varepsilon = 10^{-10} \), from which boundedness of \( \hat{\theta} \) follows. Although this \( \varepsilon \) might theoretically not be small enough, we found it sufficient for our numerical solutions.

Write out the HJB using Itô’s formula

\[
e^{-\rho(\Delta t)} V (W_{t+\Delta t}, x_{t+\Delta t}) = V (W_t, x_t) + \int \left[ -\kappa x + W \phi \left( -\kappa x + \frac{1}{2} \sigma^2 \right) V_W + \frac{1}{2} V_{x \sigma^2} + \frac{1}{2} V_{\sigma^2} \right] ds
\]

\[
+ \int [\sigma V_x + \phi V_W] dZ
\]

Conjecture the value function to be \( V = \log W + g (x) \) and plug in to get

\[
e^{-\rho(\Delta t)} V (W_{t+\Delta t}, x_{t+\Delta t}) = V (W_t, x_t) + \int (\text{ODE}) ds + \int [\sigma g' (x) + \phi] dZ
\]

We now need to show that \( \int [\sigma g' (x) + \phi (x, \hat{x})] dZ \) is a martingale. Any alternative stopping strategy can be described by a liquidation point \( x' \in [\mathbb{F} + \varepsilon, \hat{x}] \). We then know by our closed form solution from Proposition 5 that for this alternative strategy \( x' \) we have

(i) \( g' (x) \) bounded for all \( x \in [\mathbb{F} + \varepsilon, \hat{x}] \) for \( \varepsilon > 0 \).

(ii) \( \phi (x, \hat{x}) \) bounded for all \( x \in [\mathbb{F} + \varepsilon, \hat{x}] \).

Therefore, the stochastic integral is a martingale on \( [x', \hat{x}] \). On the bounded interval \( [\hat{x}, 0] \), by the boundedness of \( \phi \) due to the leverage constraint and due to our analytic solution via the variation of coefficients method, we have \( g' (x) \) bounded on the domain. Thus, the stochastic integral is a martingale.

By construction, the value function is \( C^1 \) at the optimal \( \hat{x} \). By the HJB, we know that the term (ODE) \( \leq 0 \). The optimal strategy \( \hat{x} \) will have (ODE) \( = 0 \) on the continuation region \( [\hat{x}, \hat{x}] \). Any alternative strategy \( x' \neq \hat{x} \) will by construction lead to non-positive term for (ODE) on the continuation region, and negative for any \( x' < \hat{x} \), as the institution optimally picked max \( [0, (\text{ODE})] \) as the liquidation process leads to a value of the term (ODE) of
zero. Taking expectations, it is clear that any \( x' \in [\tilde{x} + \varepsilon, \hat{x}] \) cannot do better than \( \hat{x} \).

Having solved for the necessary conditions, we can now derive sufficient conditions for the transitional point \( \hat{x} \) to be optimal. For this to be the case, we need that the value function is concave in \( \hat{x} \). We can easily derive the second order partial derivative of \( \tilde{V} \) w.r.t. \( \hat{x} \). However, this derivative still has to be evaluated numerically, as we cannot show global concavity analytically. Alternatively, and this is what we have done here, we solved for the optimal \( \hat{x} \) for given (non-optimal) transitional points \( x' \), and then optimized the value function over this range of points to find \( \hat{x} \). We know that as \( \hat{x} \to \infty \), the value function converges to the constrained value function, i.e. \( \tilde{V} \to \bar{V} \). We found unique maximum \( \hat{x} \) for each value of \( \bar{\phi} \).

\[\text{C Supplementary figures}\]

The optimal no trading region \( [\tilde{x}, \hat{x}] \) and its comparative statics are shown in figure 5. The required monetary compensation for imposing a constraint, \( \alpha \), and for withdrawing the accounting option, \( \beta \), are shown in figures 6 and 7.

**Price \( P_t \)**

![Graph of optimal no-trading region prices](image)

Figure 5: **Optimal no-trading region prices** \((P(\tilde{x}), P(\hat{x}))\) for parameters values \( \kappa = 0.8 \), \( \sigma = 0.4 \), \( \rho = 1/8 \) as a function of the constraint \( \bar{\phi} \). The solid red line depicts \( P(\hat{x}) \), the point of concealment, and the dashed red line depicts \( P(\tilde{x}) \), the point of reckoning.
Figure 6: Required proportional wealth compensation $\alpha$ for a leverage constraint w/o stale valuations at $x = 0$ for parameter values $\kappa = .8$, $\sigma = .4$, $\rho = 1/8$ as a function of the constraint $\bar{\phi}$. Panel A depicts $\alpha$ for different levels of $\kappa$: $\kappa = .8$ (solid), $\kappa = .6$ (dashed), $\kappa = 1$ (dotted). Panel B depicts $\alpha$ for different levels of $\sigma$: $\sigma = .4$ (solid), $\sigma = .3$ (dashed), $\sigma = .5$ (dotted)

Figure 7: Required proportional wealth compensation $\beta$ for loss of accounting manipulation at $x = 0$ for parameter values $\kappa = .8$, $\sigma = .4$, $\rho = 1/8$ as a function of the constraint $\bar{\phi}$. Panel A depicts $\beta$ for different levels of $\kappa$: $\kappa = .8$ (solid), $\kappa = .6$ (dashed), $\kappa = 1$ (dotted). Panel B depicts $\beta$ for different levels of $\sigma$: $\sigma = .4$ (solid), $\sigma = .3$ (dashed), $\sigma = .5$ (dotted)